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By W. Chauvenet

BINOMIAL THEOREM

AND

LOGARITHMS

FOR THE USE OF THE MIDSHIPMEN AT THE
NAVAL SCHOOL, PHILADELPHIA.

PHILADELPHIA:
PERKINS & PURVES, 134 CHESTNUT STREET.
1843.

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PREFACE.

This little work has been prepared for the use of the Midshipmen now pursuing their studies at the Naval School, Philadelphia. It was commenced with the intention of using it only "in manuscript;" but as the principal object was to embrace more practical details and a greater variety of examples than are usually given in elementary works upon Algebra, it was found necessary to enlarge it to an extent that would have rendered the use of it in that form impracticable. In preparing it for the press, the original design has been still farther extended, and the work now assumes the form of a distinct, if not a complete, treatise upon the Binomial Theorem and Logarithms. Some articles are in consequence included, which (though essential parts of a treatise upon these subjects) were not called for by the immediate wants of a class whose principal study is Navigation. Those, however, who are acquainted with the extent of the mathematics requisite in a complete theoretical exposition of Navigation, will, perhaps, not regard even these parts as altogether superfluous.

As the proper understanding of both the subjects here considered requires a correct knowledge of the nature of "powers and roots," and of "exponents in general," two short chapters under these heads are prefixed by way of introduction, and for the convenience of reference.

A rigid demonstration of the Binomial Theorem, at once simple and sufficiently elementary, has been much sought for by mathematicians. The one here given depends upon a principle which is the foundation of the Differential Calculus, and is in fact little else than a *translation* of the very simple demonstration afforded by that science into the elementary language of Algebra. It is essentially the same as that given by Bourdon, (Algébre, ed. 1837, Paris,) and by Professor Peirce of Harvard University, in his excellent treatise upon Algebra. The latter, however, has not extended the demonstration to the cases in which the exponent is negative or fractional.

The mode of deriving the logarithmic formula, in Chapter VII, is that of Euler, with some modification to render it intelligible to those not familiar with the infinitesimal calculus. Under the head "Transformations of the Logarithmic Formula," will be found some of the most expeditious formulæ given by Callet in the "Précis Elémentaire," prefixed to his "Tables Portatives," and by Borda and Delambre, in their Prefaces to the "Tables Trigonométriques Décimales."

William Chauvenet.

U. S. Naval Asylum,
Philadelphia, March, 1843.

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ERRATA.

Page 54, Art. 67, Ex. 8, for b^m read b^n.
" 81, line 10, for $-E$ read $+E$.

BINOMIAL THEOREM AND LOGARITHMS.

CHAPTER I.

POWERS AND ROOTS.

1. The student is here supposed to be acquainted with the fundamental principles of algebra; at least, with simple equations, fractions, radicals and the multiplication, division &c. of powers; but, before proceeding to the consideration of the Binomial Theorem, we shall give, in the first and second chapters, a summary view of the nature of powers and roots, and of exponents in general.

2. *The power of any quantity is that quantity multiplied by itself a certain number of times.* Thus, $a\times a$ is a power of a which is called the second power of a, because a is taken twice as a factor; and, $a\times a\times a\times a$ is the fourth power of a, because a is taken four times as a factor. But instead of repeating the letter in this manner, we indicate the operation by a small figure; thus, instead of $a\times a$, we write a^2, to show that a is repeated twice; and, instead of $a\times a\times a\times a$, we write a^4. This figure is called the *exponent* of the power.

3. *The root of any quantity is a quantity which, multiplied by itself a certain number of times, produces the first quantity.* Thus, 2 is the 3d root of 8, because $2\times2\times2=8$. That is, 8 is the 3d *power* of 2, and 2 is the 3d *root* of 8. So also, a is the 3d root of a^3; or, the 2d root of a^2; or, the 4th root of a^4, &c.

Power and *root* are correlative terms; a^m is the m^{th} power of a, and a is the m^{th} root of a^m.

The root of a quantity is expressed by means of the ***radical sign*** $\sqrt{\ }$ and an *index*. Thus, the 3d root of 8 is expressed $\sqrt[3]{8}$, the 4th root of a, $\sqrt[4]{a}$; the 5th root of x^3, $\sqrt[5]{x^3}$, &c.; $\sqrt[4]{a}$ signifies that quantity whose 4th power is a; $\sqrt[5]{x^2}$ that quantity whose 5th power is x^2, &c. Another mode of notation will be shown in the next chapter.

The power or root of a compound quantity is expressed with the aid of the vinculum or parenthesis. Thus, the 4th power of $a+b$ is expressed $(a+b)^4$; the 3d root of $x^2+y^2+z^2$, $\sqrt[3]{x^2+y^2+z^2}$, or $\sqrt[3]{}(x^2+y^2+z^2)$, &c.

4. The second power is frequently called the ***square***, the second root the ***square root;*** the third power, the ***cube***, and the third root, the ***cube root.***

The square root is denoted by the radical sign without the index. Instead of $\sqrt[2]{a}$ we write simply $\sqrt{a}$.

5. ***Powers of the same quantity are multiplied together by adding their exponents.*** Thus, $a^3\times a^2=aaa\times aa=aaaaa=a^5$, and 5 is the sum of 3 and 2.

In general $a^m\times a^n=a^{m+n}$; for $a^m=aaa\ldots\ldots$ repeated m times, and $a^n=aaa\ldots\ldots$ repeated n times, so that $a^m\times a^n=aaa\ldots\ldots$ repeated $m+n$ times, which is the $(m+n)^{\text{th}}$ power of a, or a^{m+n}. In the same way, $a^m\times a^n\times a^p\times a^q\ldots\ldots=a^{m+n+p+q}\ldots\ldots$ for any number of factors.

6. ***The division of powers of the same quantity is effected by subtracting their exponents.***

Thus, a^5 divided by a^2, is $aaaaa$ divided by aa; that is, $\frac{a^5}{a^2}=\frac{aaaaa}{aa}=aaa=a^3$, and 3 is the difference of 5 and 2.

In general $\frac{a^m}{a^n}=a^{m-n}$, for $a^m=aaa\ldots\ldots$ repeated m times, and $a^n=aaa\ldots\ldots$ repeated n times; so that $\frac{a^m}{a^n}=aaa\ldots\ldots$ repeated $m-n$ times, which is the $(m-n)^{\text{th}}$ power of a, or a^{m-n}.

7. The multiplication and division of powers of ***different*** quanti-

ties are merely represented. The product of a^2 and b^3 is a^2b^3; the quotient of a^2 divided by b^3 is $\frac{a^2}{b^3}$. Sometimes quantities to be multiplied or divided contain powers of the same quantity; these may be multiplied or divided by the above rules, and the operation upon the rest simply denoted. Thus, the product of a^2b^3c and a^3df is a^5b^3cdf; the quotient of $x^5y^6z^7$ divided by axz is $\frac{x^4y^6z^6}{a}$, and a^mb^nc divided by a^pb^qd is $\frac{a^{m-p}\,b^{n-q}c}{d}$.

8. *Involution of powers is effected by multiplying the exponents.* Thus, to involve a^2 to the 3d power, we are to take a^2 three times as a factor, which is $a^2 \times a^2 \times a^2$, or a^{2+2+2}, or a^6, (Art. 5,) that is, the 3d power of a^2 is a with the exponent 2×3.

In general the n^{th} power of a^m is a^{mn}; for $(a^m)^n = a^m \times a^m \times a^m \ldots\ldots$ repeated n times, which gives $a^{m+m+m} \ldots\ldots$ with the exponent m repeated n times; that is, with the exponent mn. Hence $(a^m)^n = a^{mn}$. So also $(a^{mn})^p = a^{mnp}$; $(x^{mnp})^q = x^{mnpq}$.

9. *The evolution of powers is effected by dividing the exponents.* Thus, to extract the square root of a^4 we divide the exponent by 2, which gives a^2. The 4th root of a^{20} is a^5; the 5th root of y^{30} is y^6, &c. The proof follows from what was shown in the last article, since evolution is the inverse of involution. In general the n^{th} root of a^{mn} is $a^{\frac{mn}{n}}$ or a^m.

10. The involution and evolution of the product or quotient of powers are effected in the same manner. The 4th power of a^2b^3 is a^8b^{12}; the 5th power of $\frac{a^3}{b^2}$ is $\frac{a^{15}}{a^{10}}$; the 4th root of $x^{12}y^4$ is x^3y; the cube root of $\frac{m^6}{n^9}$ is $\frac{m^2}{n^3}$; the cube root of $\frac{m^3n^6p^6}{x^9y^{12}}$ is $\frac{mn^2p^2}{x^3y^4}$.

11. Since $(abc \ldots.)^m = a^mb^mc^m \ldots.$; and $\sqrt[m]{a^mb^mc^m \ldots.} = abc \ldots.$, we have the following principles:—

The power of the product of any number of quantities is equal to the product of their powers; and, *the root of the product of any number of quantities is equal to the product of their roots.*

12. The root of a quantity which is not an exact power (or if a power, one whose exponent is not a multiple of the index of the root) can only be represented. For example, the 5th root of a is represented thus, $\sqrt[5]{a}$; the 4th root of a^3, $\sqrt[4]{a^3}$, &c. Hence, when it is required to extract the root of the product or quotient of several quantities, it frequently happens that we can perform the operation in respect to a part of the quantities only, and must merely represent it in the case of the others. Thus, to extract the square root of a^2b, we can take the square root of a^2, which is a, but the square root of b we must denote by the radical sign, thus, $\sqrt{b}$; so that the square root of a^2b is $a\sqrt{b}$. In the same way we find the square root of a^4b^3 to be $a^2\sqrt{b^3}$. But this last expression may be reduced to a simpler form, thus; $a^2\sqrt{b^3}=a^2\sqrt{b^2b}$, in which last form we can extract the square root b^2, but must indicate that of b, so that the expression becomes $a^2b\sqrt{b}$. In a similar manner we find,

4th root of $a^8b^{11}=\sqrt[4]{a^8b^8b^3}=a^2b^2\sqrt[4]{b^3}$.

7th root of $ab^{15}=\sqrt[7]{ab^{14}b}=b^2\sqrt[7]{ab}$.

nth root of $x^ny^m=x\sqrt[n]{y^m}$.

pth root of $\dfrac{x^py}{z^{pn}}=\dfrac{x\sqrt[p]{y}}{z^n}$.

13. *When the quantities have numeral coefficients*, the operations of involution and evolution may always be performed upon these, and the result prefixed to the literal part. Thus, the 3d root of $8a$ is $2\sqrt[3]{a}$; the square root of $16a^3b$ is $4a\sqrt{ab}$; the 5th root of $\dfrac{32a^{10}b}{c}$ is $2a^2\sqrt[5]{\dfrac{b}{c}}$.

Sometimes it is convenient to extract the root of the numeral part, even when it is not an exact power. Thus, $\sqrt{30a^2bc}=5.477a\sqrt{bc}$ nearly.

14. Thus far we have said nothing of the *signs* of powers and roots.

If we involve $-a$ to the 2d power, we have $(-a)\times(-a)$, which, by the rules of multiplication, is $+aa$ or $+a^2$; but if we involve $-a$ to the 3d power, we have $(-a)\times(-a)\times(-a)$, which, by the same rules, is $-aaa$ or $-a^3$. It is evident that the 4th power of a is $+a^4$; the 5th power of a is $-a^5$, &c.; *the even powers being positive and the uneven powers negative.* We shall thus find that $(-a)^{20}=+a^{20}$, and $(-a)^{21}=-a^{21}$, &c. &c.

15. We see then that the squares of $+a$ and $-a$ are both $+a^2$. If then it be required to extract the square root of $+a^2$, this root may be either $+a$ or $-a$; so that the result is ambiguous, which is expressed by prefixing both signs thus, $\pm a$; that is, $\sqrt{a^2}=\pm a$.

But in extracting the cube root no ambiguity will exist, for the cube of $+a$ and the cube of $-a$ are not the same, the one being $+a^3$ and the other $-a^3$; and, consequently, the cube root of $+a^3$ is $+a$, and the cube root of $-a^3$ is $-a$. In the same way the 4th root of $+a^4$ will be either $+a$ or $-a$, but the 5th root of $+a^5$ will be $+a$, and the 5th root of $-a^5$ will be $-a$.

Hence the rule, *the even root of a positive quantity is either positive or negative; the uneven root of a positive quantity is positive; and the uneven root of a negative quantity is negative.*

16. *The even root of a negative quantity is impossible.* For the square root of $-a^2$ is neither $+a$ nor $-a$; since $+a$ squared is $+a^2$, and $-a$ squared is $+a^2$; and there is no quantity which, multiplied by itself, will produce $-a^2$. For the same reason $\sqrt[4]{-a}$, $\sqrt[6]{-x^6}$, $\sqrt[8]{-1}$ are impossible. These quantities, or rather expressions, are called *imaginary*. It is usual to reduce them as follows:

To find the square root of $-a^2$, we have $\sqrt{-a^2}=\sqrt{a^2\times(-1)}=\pm a\sqrt{-1}$, (Arts. 11 and 15,) in which last form the imaginary part is the even root of -1; and to this form all imaginary expressions may be reduced. Thus,

$$\sqrt{-\frac{ab^2c}{d}}=\sqrt{\frac{ab^2c}{d}}\times\sqrt{-1},$$

in which $\sqrt{\frac{ab^2c}{d}}$ is a *possible* quantity, being the even root of a positive quantity.

17. EXAMPLES.

1. Find the product of $-x^m y^n z^p$, $x^n y^p z^m$, $-x^p y^m z^n$.
Ans. $x^{m+n+p} y^{m+n+p} z^{m+n+p}$ or $(xyz)^{m+n+p}$.

2. Divide $x^{mp} y^{np} z^{qp}$ by $x^{np} y^{qp} z^{mp}$. Ans. $(x^{m-n} y^{n-q} z^{q-m})^p$.

3. Divide $-a^{xv} b^{xz}$ by $-a^{nv} b^{nz}$. Ans. $(a^v b^z)^{x-n}$.

4. Involve $-b$ to the n^{th} power.
Ans. $+b^n$ if n is even, $-b^n$ if n is uneven.

5. Involve $-b$ to the $(2n)^{\text{th}}$ power.
Ans. $+b^{2n}$ (for $2n$ is even whatever be the value of n.)

6. Involve $-b$ to the $(2n+1)^{\text{th}}$ power.
Ans. $-b^{2n+1}$ (for whatever be the value of n, $2n$ is even, and $2n+1$ is uneven.)

7. Find the 4th root of $\frac{16x^4y^8}{xy^2}$. Ans. $\pm\frac{2xy^2}{\sqrt[4]{xy^2}}$.

8. Find the 3d root of $-\frac{a^3mn}{a^6m^4n^7}$. Ans. $-\frac{1}{amn^2}$.

9. Find the 2d root of $-\frac{ax^3y^3}{a^3xy}$. Ans. $\pm\frac{xy}{a}\sqrt{-1}$.

10. Find the n^{th} root of a^{mn}.
Ans. $+a^m$ if n is uneven, $\pm a^m$ if n is even.

11. Find the $(2n)^{\text{th}}$ root of $-a^{4n}$. Ans. $\pm a^2\sqrt[2n]{-1}$.

12. Find the $(2n)^{\text{th}}$ root of $x^{4mn}y^{2np}$. Ans. $\pm x^{2m}y^p$.

13. Find the $(2n+1)^{\text{th}}$ root of $-a$. Ans. $-\sqrt[2n+1]{a}$.

14. Find the $(2n+1)^{\text{th}}$ root of $-\frac{x^{4n+2}}{y^{8n+4}}$. Ans. $-\frac{x^2}{y^4}$.

15. Find the $(2n)^{\text{th}}$ root of $-\frac{x^{8mn}y^{4n}}{x^{6mn}y^{2n}}$. Ans. $\pm x^m y\sqrt[2n]{-1}$.

16. Find the 5th root of $1215\, a^6b^7c^8$. Ans. $3abc\sqrt[5]{5ab^2c^3}$.

CHAPTER II.

EXPONENTS IN GENERAL.

18. In the preceding chapter we have applied the rules for the multiplication, division, &c. of powers only to those cases in which the exponents were whole positive numbers. Exponents, however, may be also *fractional* and *negative*.

19. Let it be required to extract the 3d root of a^2. We have seen that this would be $\sqrt[3]{a^2}$, but algebraists have agreed to apply to such cases the general rule for the evolution of powers given in Art. 9. To extract the 3d root of a^2, according to this rule, *we divide the exponent* by 3, which gives $a^{\frac{2}{3}}$; therefore, $\sqrt[3]{a^2}$ and $a^{\frac{2}{3}}$ are equivalent expressions, both signifying "the 3d root of *a* squared;" and in the fractional form, the numerator of the fraction indicates the power, and the denominator the root. In the same way we find $\sqrt{a}=a^{\frac{1}{2}}$; $\sqrt[3]{a}=a^{\frac{1}{3}}$; $\sqrt[5]{a^9}=a^{\frac{9}{5}}$; $\sqrt[m]{a^n}=a^{\frac{n}{m}}$, &c.

The use of fractional exponents thus renders the radical sign unnecessary, and enables us to express roots and powers by the same general mode of notation.

20. Again, let it be required to divide a^5 by a^7. This is, $\frac{a^5}{a^7}$, which fraction reduced by dividing its numerator and denominator by a^5, becomes $\frac{1}{a^2}$. But by Art. 6, powers of the same quantity are divided by subtracting their exponents. Hence, to divide a^5 by a^7, we must subtract 7 from 5; but $5-7=-2$, so that the quotient will be *a* with the exponent -2, or a^{-2}. Therefore, $\frac{1}{a^2}$ and a^{-2} are equivalent expressions; but the latter has the advantage of representing a fraction in the same form with whole numbers. In the same way we shall find $\frac{1}{a^6}=a^{-6}$; $\frac{1}{a^7}=a^{-7}$; $\frac{1}{a^m}=a^{-m}$.

We see then that the negative exponent indicates the ***reciprocal of a power;*** that is, a^{-6} is the reciprocal of a^6; a^{-7} is the reciprocal of a^7; a^{-m} is the reciprocal of a^m, &c.

21. We apply the rules given in the preceding chapter also to fractional and negative exponents. Thus, to multiply $a^{\frac{1}{3}}$ by $a^{\frac{1}{2}}$ we add the exponents (Art. 5,) and we have

$$a^{\frac{1}{3}}\times a^{\frac{1}{2}}=a^{\frac{1}{3}+\frac{1}{2}}=a^{\frac{5}{6}}.$$

In the same manner we find,

$$a^{\frac{m}{n}}\times a^{\frac{p}{q}}=a^{\frac{m}{n}+\frac{p}{q}}=a^{\frac{mq+np}{nq}}.$$

$$a^{\frac{m}{n}}\times a^{-\frac{p}{q}}=a^{\frac{m}{n}-\frac{p}{q}}=a^{\frac{mq-np}{nq}}.$$

22. Applying the rule for division of powers, (Art. 6,) we have

$$\frac{a^{\frac{2}{3}}}{a^{\frac{1}{6}}}=a^{\frac{2}{3}-\frac{1}{6}}=a^{\frac{1}{2}}.$$

$$\frac{a^{\frac{m}{n}}}{a^{\frac{p}{q}}}=a^{\frac{m}{n}-\frac{p}{q}}=a^{\frac{mq-np}{nq}}.$$

$$\frac{a^{\frac{m}{n}}}{a^{-\frac{p}{q}}}=a^{\frac{m}{n}+\frac{p}{q}}=a^{\frac{mq+np}{nq}}.$$

23. Applying the rule for involution of powers, (Art. 8,) we have

$$(a^{\frac{5}{6}})^3=a^{\frac{5}{6}\times 3}=a^{\frac{5}{2}}.$$

$$(a^{\frac{m}{n}})^p=a^{\frac{m}{n}\times p}=a^{\frac{mp}{n}}.$$

$$(a^{-m})^n=a^{-mn}.$$

$$(a^m)^{-n}=a^{-mn}.$$

$$(a^{-m})^{-n}=a^{mn}.$$

24. Applying the rule for evolution of powers, (Art. 9,) we have

$$\sqrt[3]{8a^{\frac{1}{2}}}=2a^{\frac{1}{2}\div 3}=2a^{\frac{1}{6}}.$$

$$\sqrt[p]{a^{\frac{m}{n}}}=a^{\frac{m}{np}}.$$

$$\sqrt[p]{(a^m)^q}=a^{\frac{mq}{p}}.$$

25. By using fractional exponents involution and evolution are performed by the same rule; that is, by *multiplying the exponents*. Thus, the above examples will be performed,

$$\sqrt[3]{8a^{\frac{1}{2}}}=(8a^{\frac{1}{2}})^{\frac{1}{3}}\text{ (Art. 19,) }=8^{\frac{1}{3}}a^{\frac{1}{2}\times\frac{1}{3}}\text{ (Art. 8,)}=2a^{\frac{1}{6}}$$

$$\sqrt[p]{a^{\frac{m}{n}}}=(a^{\frac{m}{n}})^{\frac{1}{p}}=a^{\frac{m}{n}\times\frac{1}{p}}=a^{\frac{m}{np}}$$

$$\sqrt[p]{(a^m)^q}=(a^m)^{\frac{q}{p}}=a^{m\times\frac{q}{p}}=a^{\frac{mq}{p}}$$

In the same way we find

$$(a^{\frac{m}{n}})^{\frac{p}{q}}=a^{\frac{mp}{nq}}.$$

$$(a^{-\frac{m}{n}})^{\frac{p}{q}}=a^{-\frac{mp}{nq}}.$$

$$(a^{\frac{m}{n}})^{-\frac{p}{q}}=a^{-\frac{mp}{nq}}.$$

$$(a^{-\frac{m}{n}})^{-\frac{p}{q}}=a^{\frac{mp}{nq}}.$$

26. We conclude this chapter by exhibiting the regular series of powers of a, decreasing by unity. Beginning at a^m and dividing continually by a we have the following series:

$$a^m,\ a^{m-1},\ a^{m-2}\ldots a^3,\ a^2,\ a^1,\ a^0,\ a^{-1},\ a^{-2},\ a^{-3},\ldots\ a^{-m+2},\ a^{-m+1},\ a^{-m}.$$

which is a regular series of powers of a from $+m$ to $-m$. Any two terms equally distant from a^0 are the reciprocals of each other; thus, a^1 is the reciprocal of a^{-1}, a^2 of a^{-2}, &c., and a^m of a^{-m}.

The term a^0 is found (as the other terms are found) by subtracting 1 from the exponent of the preceding term, which is equivalent to dividing by a or a^1, (Art. 6.)

$$\text{Thus, } \frac{a^1}{a^1}=a^{1-1}=a^0. \quad \text{But } \frac{a^1}{a^1} \text{ or } \frac{a}{a}=1,$$

since the quotient obtained by dividing any quantity by itself is unity. Hence the singular result,

$$a^0=1,$$

which is true, independently of the value of a; so that if $a=1, 2, 3, 4,$ &c., we shall have,

$$1^0=1,\ 2^0=1,\ 3^0=1,\ 4^0=1, \text{ \&c.}$$

CHAPTER III.

BINOMIAL THEOREM.

27. The rules for involution and evolution already given apply only to monomials, or quantities consisting of a single term. *Binomials*, or quantities consisting of two terms, connected by the signs + or —, may be involved by actual multiplication. Thus, the powers of $a+b$ may be found by actually multiplying it by itself, as follows:

$$\begin{array}{rl}
 & a+b \\
 & a+b \\ \hline
 & a^2+ab \\
 & \qquad ab+b^2 \\ \hline
(a+b)^2= & a^2+2ab+b^2 \\
 & a+b \\ \hline
 & a^3+2a^2b+ab^2 \\
 & \qquad a^2b+2ab^2+b^3 \\ \hline
(a+b)^3= & a^3+3a^2b+3ab^2+b^3
\end{array}$$

In this way we may proceed to find any power of $a+b$, but the process will evidently be very tedious for high powers. Some general rule is therefore desirable, which will enable us to express any power of a binomial without performing the multiplication. On the other hand, some rule or formula is desirable, by which we can express any root of a binomial. We have seen, in the case of monomials, that fractional and negative exponents may be treated by the same rules as positive and integral ones; and (Art. 25,) that with the use of fractional exponents the same rule effects both involution and evolution. We might infer that the same generality exists for the powers and roots of binomials, and that the same rule, or formula, by which we can express any power, will also serve to express any root. In fact, such a formula is *Newton's Binomial Theorem*, by which we can express the value of $(a+b)^m$, whether m be *positive, negative, integral or fractional.*

Thus, if m is negative or $=-n$, this theorem enables us to express the value of $(a+b)^{-n}$ or (Art. 20,) of $\frac{1}{(a+b)^n}$.

If $m=\frac{1}{p}$, it expresses the value of $(a+b)^{\frac{1}{p}}$, or (Art. 19,) of $\sqrt[p]{a+b}$.

If $m=-\frac{1}{p}$, it expresses the value of $(a+b)^{-\frac{1}{p}}$, or (Art. 19, 20,) of $\frac{1}{\sqrt[p]{a+b}}$.

It is in these applications that the great utility of this theorem is manifested.

We shall here demonstrate it in the most general manner, and in the next chapter give the applications to the particular cases. We must first establish the following important principles.

INDETERMINATE COEFFICIENTS.

28. *Whenever we have an equation of the form,*

$$A+Bx+Cx^2+Dx^3+\text{ \&c.}=A'+B'x+C'x^2+D'x^3+\text{ \&c.},$$

which is true whatever be the value of x, then, the coefficients of the like powers of x in the two members are equal each to each, and we shall have,

$$A=A',\ B=B',\ C=C',\ \text{\&c.}$$

For since the equation is true whatever be the value of x, it is true for the value, $x=0$. But if this value is substituted for x, all the terms become 0, except A and A', which being independent of x, that is, not containing x in any form, do not become 0. Therefore the equation is reduced to

$$A=A'.$$

These quantities, therefore, destroy each other in the original equation, which then becomes

$$Bx+Cx^2+Dx^3+\text{ &c.}=B'x+C'x^2+D'x^3+\text{ &c.}$$

Divide this by x, and we have

$$B+Cx+Dx^2+\text{ &c.}=B'+C'x+D'x^2+\text{ &c.}$$

which must also be true when $x=0$; but in that case all the terms become 0 except B and B', and we have simply

$$B=B'.$$

Our equation, therefore, is

$$Cx+Dx^2+\text{ &c.}=C'x+D'x^2+\text{ &c.}$$

And it is shown in precisely the same manner, that

$$C=C',\ D=D',\ E=E',\text{ &c.}$$

29. It follows from this, that if we have an equation of the form

$$A-A'+(B-B')x+(C-C')x^2+(D-D')x^3+\text{ &c.}=0,$$

which is true for any value of x, the coefficients must all be equal to 0; for this equation is derived from the one proposed in the preceding article, by transposing all the terms to the left hand member, and it has been shown, that $A=A'$, $B=B'$, &c., whence $A-A'=0$, $B-B'=0$, &c.

And if we put $A-A'=M$, $B-B'=N$, &c., the equation will be,

$$M+Nx+Px^2+Qx^3+\text{ &c.}=0.$$

Whenever, then, we have an equation of the form,

$$M+Nx+Px^2+Qx^3+\text{ &c.}=0,$$

which is true for any value of x, all the coefficients are equal to 0, and we shall have,

$$M=0,\ N=0,\ P=0,\text{ &c.}$$

30. This principle, simple as it is, is one of the most prolific in algebra. It is the distinguishing feature of algebra, that its processes are *general*, and embrace large classes of particular cases. Hence it is often employed to investigate *formulæ* which shall have the same form, whatever be the value of certain of the quantities which enter into them; as in the case of the binomial theorem, it is employed in discovering a formula for expressing any power or root of $a+b$, which formula shall be the same whatever are the values of a and b, and whatever the power or root sought. It is in such investigations that the principle above demonstrated is particularly useful. A series is *assumed* to be the required formula, but the coefficients are all unknown quantities, and are consequently called ***indeterminate coefficients.*** If an equality can then be shown to exist between this series and another of the same form, the above principle gives us as many equations as there are coefficients, from which the values of the coefficients are then determined by elimination. These remarks will be clearly comprehended by attending to the demonstration given in Art. 33.

OF THE QUOTIENT OF $\frac{x^m-y^m}{x-y}$.

31. ***The difference of two powers of the same degree, x^m-y^m, is exactly divisible by the difference of their roots, $x-y$, if m is a positive integer.***

The process of division will be,

$$\begin{array}{r|l}
 & x^m-y^m \quad \\
 & \underline{x^m-x^{m-1}y} \\
\text{1st rem.} = & x^{m-1}y-y^m \\
 & \underline{x^{m-1}y-x^{m-2}y^2} \\
\text{2d rem.} = & x^{m-2}y^2-y^m
\end{array}
\qquad
\begin{array}{|l}
x-y \\
\hline
x^{m-1}+x^{m-2}y
\end{array}$$

Now it is evident, that by continuing this process, the next term of the quotient will be found to be $x^{m-3}y^2$, and that the 3d remainder will be $x^{m-3}y^3-y^m$; the 4th remainder $x^{m-4}y^4-y^m$, &c.; so that the m^{th} remainder will be $x^{m-m}y^m-y^m$. But $x^{m-m}=x^0=1$, (Art. 26,) therefore, this last remainder is $y^m-y^m=0$; that is, the process terminates at the m^{th} term, since the remainder is equal to 0; and the division is consequently exact.

The last term of the quotient may be obtained by considering the $(m-1)$th remainder, which will be $xy^{m-1}-y^m$, which divided by $x-y$ gives y^{m-1}. In the same way, from the $(m-2)$nd remainder the last term but one of the quotient is found to be xy^{m-2}. The form of the quotient is therefore evident, and we have

$$\frac{x^m-y^m}{x-y}=x^{m-1}+x^{m-2}y+x^{m-3}y^2+\dots\dots+x^2y^{m-3}+xy^{m-2}+y^{m-1}.$$

The number of terms in this quotient is equal to m; for y is contained in all the terms except the first, and the exponents of y are 1, 2, 3, 4, &c. to $m-1$; so that the number of terms containing y is $m-1$, and the whole number of terms, including the first, is $m-1+1$ or m.

32. *The value of* $\frac{x^m-y^m}{x-y}$ *when* $x=y$ *is in all cases* mx^{m-1}, *whether* m *is positive, negative, integral or fractional.*

The value of this quotient, when m is a positive integer, has been found in the last article to be

$$\frac{x^m-y^m}{x-y}=x^{m-1}+x^{m-2}y+x^{m-3}y^2+\dots\dots+x^2y^{m-3}+xy^{m-2}+y^{m-1}.$$

When $x=y$, this equation becomes

$$\frac{x^m-x^m}{x-x}=x^{m-1}+x^{m-1}+x^{m-1}+\dots\dots+x^{m-1}+x^{m-1}+x^{m-1}.$$

But the number of terms in the right hand member has been shown to be m, therefore we have

$$\frac{x^m-x^m}{x-x}=mx^{m-1};$$

so that our proposition is true when m is a positive integer.

To extend it to the other cases, FIRST, *let* m *be a negative integer, or let* $m=-n$.

Then we have

$$\frac{x^m-y^m}{x-y}=\frac{x^{-n}-y^{-n}}{x-y}=\frac{x^{-n}y^{-n}(y^n-x^n)}{x-y}=\frac{-x^{-n}y^{-n}(x^n-y^n)}{x-y}.$$

But n is a positive integer; therefore, by what has just been shown, when $x=y$

$$\frac{x^n-y^n}{x-y}=\frac{x^n-x^n}{x-x}=nx^{n-1}.$$

Also, $x^{-n}y^{-n}=x^{-n}x^{-n}=x^{-2n}$, (Art. 5.)

Therefore, when $x=y$, we have

$$\frac{x^{-n}-y^{-n}}{x-y}=-x^{-n}y^{-n}\times\frac{x^{n}-y^{n}}{x-y}=-x^{-2n}nx^{n-1}=-nx^{-n-1},$$

or restoring m,

$$\frac{x^{m}-x^{m}}{x-x}=mx^{m-1},$$ m being a negative integer.

SECOND. *Let m be a positive fraction, or* $m=\frac{p}{q}$.

Then, $$\frac{x^{m}-y^{m}}{x-y}=\frac{x^{\frac{p}{q}}-y^{\frac{p}{q}}}{x-y}.$$

Let $x^{\frac{1}{q}}=v$, and $y^{\frac{1}{q}}=u$; then substituting we have

$$\frac{x^{\frac{p}{q}}-y^{\frac{p}{q}}}{x-y}=\frac{v^{p}-u^{p}}{v^{q}-u^{q}}=\frac{\frac{v^{p}-u^{p}}{v-u}}{\frac{v^{q}-u^{q}}{v-u}},$$ (dividing both numerator and denominator by $v-u$.) But p and q are positive integers; therefore, when $x=y$ and consequently $v=u$, we have

$$\frac{v^{p}-u^{p}}{v-u}=\frac{v^{p}-v^{p}}{v-v}=pv^{p-1}=px^{\frac{p}{q}-\frac{1}{q}},$$

$$\frac{v^{q}-u^{q}}{v-u}=\frac{v^{q}-v^{q}}{v-v}=qv^{q-1}=qx^{1-\frac{1}{q}},$$

which values substituted give

$$\frac{x^{\frac{p}{q}}-y^{\frac{p}{q}}}{x-y}=\frac{px^{\frac{p}{q}-\frac{1}{q}}}{qx^{1-\frac{1}{q}}}=\frac{p}{q}x^{\frac{p}{q}-1},$$ (Art. 6,)

or restoring m,

$$\frac{x^{m}-x^{m}}{x-x}=mx^{m-1},$$ m being a positive fraction.

THIRD. *Let m be a negative fraction, or* $m=-\frac{p}{q}$.

Then $$\frac{x^{m}-y^{m}}{x-y}=\frac{x^{-\frac{p}{q}}-y^{-\frac{p}{q}}}{x-y}.$$

Let $x^{\frac{1}{q}}=v$, and $y^{\frac{1}{q}}=u$; then $x^{-\frac{p}{q}}=v^{-p}$, $y^{-\frac{p}{q}}=u^{-p}$, $x=v^q$, and $y=u^q$, (Art. 8,) which values substituted give

$$\frac{x^{-\frac{p}{q}}-y^{-\frac{p}{q}}}{x-y}=\frac{v^{-p}-u^{-p}}{v^q-u^q}=\frac{\frac{v^{-p}-u^{-p}}{v-u}}{\frac{v^q-u^q}{v-u}},$$ (dividing both numerator and denominator by $v-u$.) But $-p$ is a negative integer, and q is a positive integer; therefore, by what has already been shown, when $x=y$ and consequently $v=u$, we have

$$\frac{v^{-p}-u^{-p}}{v-u}=\frac{v^{-p}-v^{-p}}{v-v}=-pv^{-p-1}=-px^{-\frac{p}{q}-\frac{1}{q}}.$$

$$\frac{v^q-u^q}{v-u}=\frac{v^q-v^q}{v-v}=qv^{q-1}=qx^{1-\frac{1}{q}}.$$

Substituting these values, we have

$$\frac{x^{-\frac{p}{q}}-y^{-\frac{p}{q}}}{x-y}=\frac{-px^{-\frac{p}{q}-\frac{1}{q}}}{qx^{1-\frac{1}{q}}}=-\frac{p}{q}x^{-\frac{p}{q}-1},$$

or restoring m,

$$\frac{x^m-x^m}{x-x}=mx^{m-1},$$ m being a negative fraction.

DEMONSTRATION OF THE BINOMIAL THEOREM.

33. *It is required to obtain a general formula expressing the value of* $(a+b)^m$, *whether m be integral, fractional, positive or negative.* In other words, it is required to obtain the general development of $(a+b)^m$.

In order to simplify, we resolve the binomial into two factors, thus;

$$(a+b)^m=\left[a(1+\frac{b}{a})\right]^m=a^m\left(1+\frac{b}{a}\right)^m, \text{(Art. 11.)}$$

If then we obtain the development of $\left(1+\frac{b}{a}\right)^m$, we have only to multiply it by a^m to obtain that of $(a+b)^m$.

Let $\frac{b}{a}=x$, then to develop $(1+x)^m$ assume

$$(1+x)^m=A+Bx+Cx^2+Dx^3+\text{ \&c.}, \tag{1}$$

in which A, B, C, D, &c. are indeterminate coefficients, independent of x, and we are now to determine their values.*

Now this equation must be true for any value of x; therefore, let $x=0$, and we have

$$(1)^m=A, \text{ or } A=1.$$

Substituting this value in (1), we have

$$(1+x)^m=1+Bx+Cx^2+Dx^3+\text{ \&c.} \tag{2}$$

Again, since the development is to have the same form for all values of x, let $x=y$; then (2) becomes

$$(1+y)^m=1+By+Cy^2+Dy^3+\text{ \&c.} \tag{3}$$

Subtract (3) from (2),

$$(1+x)^m-(1+y)^m=B(x-y)+C(x^2-y^2)+D(x^3-y^3)+\text{ \&c.}$$

Divide this by $x-y$,

$$\frac{(1+x)^m-(1+y)^m}{x-y}=B+C.\frac{x^2-y^2}{x-y}+D.\frac{x^3-y^3}{x-y}+\text{ \&c.} \tag{4}$$

Let $1+x=v$, $1+y=u$; then $x-y=v-u$, which values substituted in the left hand member of (4), give

$$\frac{v^m-u^m}{v-u}=B+C.\frac{x^2-y^2}{x-y}+D.\frac{x^3-y^3}{x-y}+\text{ \&c.} \tag{5}$$

But when $x=y$, $v=u$, and we have, by Art. 32, whatever be the value of m, positive, negative, integral or fractional,

$$\frac{v^m-u^m}{v-u}=\frac{v^m-v^m}{v-v}=mv^{m-1}=m(1+x)^{m-1}.$$

And we have also for the quotients in the right hand member of (5), when $x=y$, by Art. 32,

$$\frac{x^2-y^2}{x-y}=\frac{x^2-x^2}{x-x}=2x^{2-1}=2x, \quad \frac{x^3-y^3}{x-y}=3x^{3-1}=3x^2, \text{ \&c.}$$

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* The coefficients A, B, C, D, &c., being indefinite, the expression $A+Bx+Cx^2+$ &c., may be assumed to be equal to any quantity whatever. It is then to be shown what values these coefficients must have, in order to satisfy the equation.

These values substituted in (5), give

$$m(1+x)^{m-1}=B+2Cx+3Dx^2+4Ex^3+\text{ \&c.} \qquad (6)$$

Multiplying this equation by $1+x$, we have

$$\begin{aligned} m(1+x)^m &= \left.\begin{aligned} &B+2Cx+3Dx^2+4Ex^3+\text{ \&c.} \\ &\quad+Bx+2Cx^2+3Dx^3+\text{ \&c.} \end{aligned}\right\} \\ &=B+(2C+B)x+(3D+2C)x^2+(4E+3D)x^3+\text{ \&c.} \qquad (7) \end{aligned}$$

But if we multiply (2) by m, we have

$$m(1+x)^m=m+mBx+mCx^2+mDx^3+\text{ \&c.} \qquad (8)$$

Forming an equation from (7) and (8), we obtain

$$\begin{aligned} &B+(2C+B)x+(3D+2C)x^2+(4E+3D)x^3+\text{ \&c.}= \\ &\qquad m+mBx+mCx^2+mDx^3+\text{ \&c.} \qquad (9) \end{aligned}$$

But this equation is of the form treated of in Art. 28; for its coefficients are quantities independent of x, and the equation is true for all values of x. Therefore, the coefficients of the like powers of x are equal each to each, and we have

$$\begin{array}{lll} B=m & \text{whence } B & = \quad m \\ 2C+B=mB & \text{"} \quad C=\dfrac{mB-B}{2}=\dfrac{B(m-1)}{2}=\dfrac{m(m-1)}{2}, & \\ 3D+2C=mC & \text{"} \quad D=\dfrac{mC-2C}{3}=\dfrac{C(m-2)}{3}=\dfrac{m(m-1)(m-2)}{2\ .\ 3}, & \\ 4E+3D=mD & \text{"} \quad E=\dfrac{mD-3D}{4}=\dfrac{D(m-3)}{4}=\dfrac{m(m-1)(m-2)(m-3)}{2\ .\ 3\ .\ 4}. & \\ \text{\&c.} & \qquad\qquad \text{\&c.} \qquad\qquad \text{\&c.} & \end{array}$$

These values of B, C, D, E, &c. substituted in (2), give

$$\begin{aligned} (1+x)^m=1+mx+\frac{m(m-1)}{2}x^2+\frac{m(m-1)(m-2)}{2\ .\ 3}x^3+ \\ \frac{m(m-1)(m-2)(m-3)}{2\ .\ 3\ .\ 4}x^4+\text{ \&c.} \qquad (10) \end{aligned}$$

which is the general development of $(1+x)^m$. The *law* of the series is evident, and any number of terms may be written at pleasure. It only remains to find the development of $(a+b)^m$. In (10) restore the value of $x=\frac{b}{a}$, we have

$$\left(1+\frac{b}{a}\right)^m=1+m.\frac{b}{a}+\frac{m(m-1)}{2}.\frac{b^2}{a^2}+\frac{m(m-1)(m-2)}{2\ .\ 3}.\frac{b^3}{a^3}+\text{ \&c.}$$

Finally, multiplying by a^m, we obtain

$$(a+b)^m = a^m + ma^{m-1}b + \frac{m(m-1)}{2}a^{m-2}b^2 + \frac{m(m-1)(m-2)}{2 \,.\, 3}a^{m-3}b^3 + \text{\&c.} \qquad (11)$$

which is the general formula for the development of any binomial $(a+b)^m$, whatever be the values of a and b, and whether m be positive, negative, integral or fractional; and this formula is known as the *Binomial Theorem of Sir Isaac Newton.*

34. If in formula (10) x be made negative, the uneven powers of x will be negative and the even ones positive, (Art. 14), so that the formula will become

$$(1-x)^m = 1 - mx + \frac{m(m-1)}{2}x^2 - \frac{m(m-1)(m-2)}{2 \,.\, 3}x^3 + \text{\&c.} \qquad (12)$$

Formulæ (10) and (12) may be expressed in one formula, thus;

$$(1 \pm x)^m = 1 \pm mx + \frac{m(m-1)}{2}x^2 \pm \frac{m(m-1)(m-2)}{2 \,.\, 3}x^3 + \text{\&c.} \qquad (13)$$

35. If we make b negative in (11) we shall find in the same manner

$$(a-b)^m = a^m - ma^{m-1}b + \frac{m(m-1)}{2}a^{m-2}b^2 - \text{\&c.} \qquad (14)$$

Expressing (11) and (14) in the same formula,

$$(a \pm b)^m = a^m \pm ma^{m-1}b + \frac{m(m-1)}{2}a^{m-2}b^2 \pm \text{\&c.} \qquad (15)$$

Formula (13) is only a particular case of (15) when $a=1$ and $b=x$.

36. In assuming our series we included all the powers of x, from 0 upwards; so that the above developments are series of an *infinite* number of terms. But when m is a positive integer, the series will terminate at the $(m+1)^{\text{th}}$ term, and all the succeeding terms will vanish, or become $=0$. For by examining formula (15), it will be seen that the 2d term contains the factor m, the 3d term the factor $m-1$, the 4th term the factor $m-2$, &c., and the $(m+2)^{\text{nd}}$ term will contain the factor $m-m$ or 0, which reduces that term to 0;

and since all the succeeding terms also contain the same factor, they all become 0. There will, therefore, remain $m+1$ terms only.

The literal part of the $(m+1)^{\text{th}}$ term will be $a^{m-m}b^m=a^0b^m=b^m$, (Art. 26,) and the coefficient will be

$$\frac{m(m-1)(m-2)\ldots\ldots\ldots\ldots(m-(m-2))(m-(m-1))}{2\,.\,3\,.\,4\quad\ldots\ldots\quad(m-2)(m-1)m},$$

or reversing the order of the factors in the denominator

$$\frac{m(m-1)(m-2)\ldots\ldots\ldots\ldots2\,.\,1}{m(m-1)(m-2)\ldots\ldots\ldots2}=1.$$

Hence the $(m+1)^{\text{th}}$ term, or last term, is simply b^m, which is analogous to the first term a^m. It might be shown in the same manner that the term preceding the last is mab^{m-1}, and in general, that ***the terms equally remote from the first and last have equal coefficients.*** But this appears more readily by taking the development of $(b+a)^m$, which is of course identical with that of $(a+b)^m$. We have at once

$$(b+a)^m=b^m+mb^{m-1}a+\frac{m(m-1)}{2}b^{m-2}a^2+\ \&c.$$

37. When m is not a positive integer it is evident that no one of the factors

$$m,\ m-1,\ m-2,\ m-3,\ \&c.$$

can be equal to 0; so that the development will, in that case, be an infinite series.*

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* This celebrated theorem is generally ascribed to Newton, but Hutton, in his History of Logarithms, shows that it was used in a particular form by several mathematicians before the time of Newton. He mentions Lucas de Burgo, who, as early as 1470, extracted the cube root by the coefficients of the power of a binomial, but who did not go to any higher roots. Hutton proceeds; "And this is the first mention I have seen made of this law of the coefficients of the powers of a binomial, commonly called Sir Isaac Newton's binomial theorem, though it is very evident that Sir Isaac was not the first inventor of it: the part of it properly belonging to him seems to be only the extending it to fractional indices, which was indeed an immediate effect of the general method of denoting all roots like powers with fractional exponents, the theorem being not at all altered. However, it appears that our author Briggs was the first who taught the rule for generating the coefficients of the terms successively one from

CHAPTER IV.

APPLICATION OF THE BINOMIAL THEOREM.

38. The most simple application of the binomial theorem is the expansion of a binomial with a positive integral exponent. To take the most simple case, let us find the development of $(a+b)^2$. Making $m=2$ in (11), we have

$$(a+b)^2=a^2+2ab+\frac{2(2-1)}{2}a^0b^2+\frac{2(2-1)(2-2)}{2\ .\ 3}a^{-1}b^3+\ \&c.$$

Here the coefficient of the 3d term is $\frac{2.1}{2}$ or 1, and $a^0=1$; therefore, this term is simply b^2. The coefficient of the 4th term is $\frac{2.1.0}{2.3}=\frac{2}{6}\times 0=0$; therefore, this term is equal to 0, and all the succeeding terms are equal to 0, since they all contain the factor 2—2 or 0. Therefore, our development contains but three terms, or

$$(a+b)^2=a^2+2ab+b^2,$$

which agrees with the result obtained by the actual multiplication of $(a+b)(a+b)$.

The 3d power of $a+b$ will be found from the formula in the same manner; by making $m=3$, we shall find

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3.$$

The 5th term, and all the succeeding terms vanish, and there remain only four terms, or $m+1$ terms, as was shown in Art. 36.

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another, of any power of a binomial, independent of those of any other power." It seems, nevertheless, that Newton was not acquainted with what had been done by Briggs. In another place Hutton makes the following remark: "But I do not wonder that Briggs' remark was unknown to Newton, who owed almost every thing to genius and deep meditation, but very little to reading: and I have no doubt that he made the discovery himself, without any light from Briggs, and that he thought it was new for all powers in general, as it was indeed for roots and quantities with fractional and irrational exponents."—*Hutton's Introduction to Mathematical Tables.*

39. To find the 6th power of $a+b$, in (11) make $m=6$; then

$$(a+b)^6=a^6+6a^5b+15a^4b^2+20a^3b^3+ \&c.$$

Here we have found the first half of the coefficients, but it is unnecessary to find the remaining ones by the formula, since, as was shown in Art. 36, *the coefficients of terms equally distant from the first and last terms are equal to each other.* Therefore the remaining coefficients will be the same as those already found, but taken in the reverse order. Hence the complete expansion is

$$(a+b)^6=a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6.$$

The literal part of the expression is easily put down, observing in all cases that the exponents of a commence at the highest in the first term, and decrease regularly by 1, and that those of b commence in the second term with 1, and .increase regularly by 1 to the highest.

The expansion of $(a-b)^6$ is found in the same manner from (14) to be

$$(a-b)^6=a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6.$$

40. Any binomial may be involved by substituting in our formulæ, for a, b and m. For example, required the 5th power of $2x-3y$. In (14) substitute for a, b and m the values $a=2x$, $b=3y$, $m=5$, and we obtain

$$(2x-3y)^5=(2x)^5-5(2x)^4(3y)+10(2x)^3(3y)^2-10(2x)^2(3y)^3+5(2x)(3y)^4-(3y)^5,$$

$$=32x^5-240x^4y+720x^3y^2-1080x^2y^3+810xy^4-243y^5.$$

EXAMPLES.

1. Find the 7th power of $1\pm 2y$.

In (13) put $x=2y$, $m=7$, then

$$(1\pm 2y)^7=1\pm 14y+84y^2\pm 280y^3+560y^4\pm 672y^5+448y^6\pm 128y^7.$$

2. Find the 4th power of $2+4xy$. We have

$$(2+4xy)^4=2^4(1+2xy)^4=16(1+2xy)^4.$$

$$(1+2xy)^4=1+8xy+24x^2y^2+32x^3y^3+16x^4y^4,$$

$$(2+4xy)^4=16(1+2xy)^4=16+128xy+384x^2y^2+512x^3y^3+256x^4y^4.$$

3. Find the 4th power of $3a^2c-2bd$.

Ans. $81a^8c^4-216a^6c^3bd+216a^4c^2b^2d^2-96a^2cb^3d^3+16b^4d^4$.

41. *Expansion of binomials with negative integral exponents.* This is effected in the same way as in the case of positive integral exponents, by substitution in formulæ (13) and (15), these formulæ having been proved true for any value of the exponent. Therefore, to expand $(a+b)^{-1}$, or (Art. 20,) $\frac{1}{a+b}$, in (15) let $m=-1$, and we have

$$(a+b)^{-1}=a^{-1}-a^{-2}b+a^{-3}b^2-a^{-4}b^3+\text{ \&c.},$$

or

$$\frac{1}{a+b}=\frac{1}{a}-\frac{b}{a^2}+\frac{b^2}{a^3}-\frac{b^3}{a^4}+\text{ \&c.}$$

which is, by Art. 37, an infinite series. The same result will be obtained by actual division:

$$\begin{array}{l|l}
1 & a+b \\ \cline{2-2}
1+\frac{b}{a} & \frac{1}{a}-\frac{b}{a^2}+\frac{b^2}{a^3}-\text{ \&c.} \\ \cline{1-1}
-\frac{b}{a} & \\
-\frac{b}{a}-\frac{b^2}{a^2} & \\ \cline{1-1}
\frac{b^2}{a^2} & \\
\frac{b^2}{a^2}+\frac{b^3}{a^3} & \\ \cline{1-1}
\text{\&c.} &
\end{array}$$

from which it is also evident that the series will never terminate.

To convert the fraction $\frac{1}{1+x}$ into an infinite series; in other words, to expand $(1+x)^{-1}$.

In (13) let $m=-1$. Then we have

$$(1+x)^{-1}=1-x+x^2-x^3+\text{ \&c.}$$

or

$$\frac{1}{1+x}=1-x+x^2-x^3+\text{ \&c.}$$

If in this equation we make $x=1$, it becomes

$$\frac{1}{2}=1-1+1-1+\text{ \&c.}$$

This singular series is only the limit of a large class of series which are *divergent*, or whose terms instead of constantly decreasing go on increasing to infinity; so that the sum of any number of terms can never express the value of the series. In the above example the sum of 2 terms is 0, of 3 terms 1, of 4 terms 0, &c., from which nothing more can be inferred than that the value of the series is between 0 and 1.*

42. A series, however, is readily obtained which correctly expresses the value of the fraction $\frac{1}{2}$. Take the fraction $\frac{1}{x+\frac{1}{2}}$, or $\left(x+\frac{1}{2}\right)^{-1}$, and in (15) let $a=x$, $b=\frac{1}{2}$, $m=-1$; we have

$$\left(x+\frac{1}{2}\right)^{-1}=x^{-1}-\frac{x^{-2}}{2}+\frac{x^{-3}}{4}-\frac{x^{-4}}{8}+\text{ \&c.}$$

or
$$\frac{1}{x+\frac{1}{2}}=\frac{1}{x}-\frac{1}{2}\left(\frac{1}{x}\right)^2+\frac{1}{4}\left(\frac{1}{x}\right)^3-\frac{1}{8}\left(\frac{1}{x}\right)^4+\text{ \&c.}$$

Now let $x=\frac{3}{2}$, then $\frac{1}{x}=\frac{2}{3}$, and we have

$$\frac{1}{2}=\frac{2}{3}-\frac{1}{2}\cdot\frac{4}{9}+\frac{1}{4}\cdot\frac{8}{27}-\frac{1}{8}\cdot\frac{16}{81}+\text{ \&c.}$$

---

* The distinguished LEIBNITZ, however, according to LAPLACE, imagined that the form of this series warranted the deduction from it of the value $\frac{1}{2}$. He observed, "that the series becomes 0 or 1, according as we take an even or an uneven number of terms; and as in infinity there is no reason for preferring the even to the uneven number, we should, according to the rules of probabilities, take the *mean*, which gives $\frac{1}{2}$ for the value of the series." But this reasoning is fallacious; for if the fraction $\frac{1+x}{1+x+x^2}$ be converted into a series, it will be found to be

$$1-x^2+x^3-x^5+x^3-\text{ \&c.,}$$

and when $x=1$, the fraction becomes $\frac{2}{3}$ and the series $1-1+1-1+$ &c. It appears, therefore, that the value of this series is indefinite, and that it may correspond to various fractions between 0 and 1.—*Laplace, Essai Philosophique sur les Probabilités.*

$$=\frac{2}{3}-\frac{2}{9}+\frac{2}{27}-\frac{2}{81}+\text{ \&c.}$$

The sum of 4 terms of this series is $\frac{40}{81}$, which differs from $\frac{1}{2}$ by $\frac{1}{162}$; the sum of 6 terms is $\frac{364}{729}$, which differs from $\frac{1}{2}$ by only $\frac{1}{1458}$; and the greater the number of terms we add together, the nearer we approximate to the value $\frac{1}{2}$, which is therefore the value of an infinite number of terms. A series of this kind, in which the terms decrease to infinity, is called a *convergent* series.

43. If in (13) we make $m=-n$, we shall find

$$(1\pm x)^{-n}=1\mp nx+\frac{(-n)(-n-1)}{2}x^2\pm\frac{(-n)(-n-1)(-n-2)}{2\,.\,3}x^3+\text{ \&c.}$$

or reducing

$$(1\pm x)^{-n}=1\mp nx+\frac{n(n+1)}{2}x^2\mp\frac{n(n+1)(n+2)}{2\,.\,3}x^3+\text{ \&c.}\qquad(16)$$

If in (15) we make $m=-n$, we shall find in the same manner

$$(a\pm b)^{-n}=a^{-n}\mp na^{-n-1}b+\frac{n(n+1)}{2}a^{-n-2}b^2\mp$$

$$\frac{n(n+1)(n+2)}{2\,.\,3}a^{-n-3}b^3+\text{ \&c.}$$

or by Art. 20,

$$\frac{1}{(a\pm b)^n}=\frac{1}{a^n}\left(1\mp n\frac{b}{a}+\frac{n(n+1)}{2}\cdot\frac{b^2}{a^2}\mp\frac{n(n+1)(n+2)}{2\,.\,3}\cdot\frac{b^3}{a^3}+\text{\&c.}\right)(17)$$

These two formulæ (16) and (17) may be used to expand binomials with negative integral exponents instead of the general formulæ; and will be found much more convenient in practice.

44. Thus, to expand $\frac{y}{(3x+y)^2}$ or $y\times\frac{1}{(3x+y)^2}$. By (17), making $a=3x$, $b=y$, $n=2$, we find

$$\frac{1}{(3x+y)^2}=\frac{1}{(3x)^2}\left(1-2\left(\frac{y}{3x}\right)+\frac{2.3}{2}\left(\frac{y}{3x}\right)^2-\frac{2.3.4}{2.3}\left(\frac{y}{3x}\right)^3+\text{ \&c.}\right)$$

multiplying by y and reducing

$$\frac{y}{(3x+y)^2}=\frac{y}{9x^2}\left(1-\frac{2y}{3x}+\frac{3y^2}{9x^2}-\frac{4y^3}{27x^3}+\frac{5y^4}{81x^4}-\text{ \&c.}\right)$$

EXAMPLES.

1. $\frac{1}{(2x^3-y^3)^2}=\frac{1}{4x^6}\left(1+\frac{2y^3}{2x^3}+\frac{3y^6}{4x^6}+\frac{4y^9}{8x^9}+\frac{5y^{12}}{16x^{12}}+\text{ \&c.}\right)$

2. $\frac{1}{1-x}=1+x+x^2+x^3+x^4+x^5+x^6+x^7+\text{ \&c.}$

When $x=\frac{1}{2}$ this becomes

$$2=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots.\text{ \&c.}$$

3. $\frac{1}{(1+2y)^5}=1-10y+60y^2-280y^3+\text{ \&c.}$

4. $\frac{ax}{a^2-x^2}=\frac{x}{a}\left(1+\frac{x^2}{a^2}+\frac{x^4}{a^4}+\frac{x^6}{a^6}+\text{ \&c.}\right)$

5. $\frac{a^2x^2}{(a\pm x)^4}=\frac{x^2}{a^2}\left(1\mp\frac{4x}{a}+\frac{10x^2}{a^2}\mp\frac{20x^3}{a^3}+\frac{35x^4}{a^4}\mp\text{ \&c.}\right)$

45. *Expansion of binomials with positive fractional exponents.* It will be convenient here, as in the last case, to obtain a formula for fractional exponents.

In (13) let $m=\frac{p}{n}$, we find

$$(1\pm x)^{\frac{p}{n}}=1\pm\frac{p}{n}x+\frac{\frac{p}{n}\left(\frac{p}{n}-1\right)}{2}x^2\pm\frac{\frac{p}{n}\left(\frac{p}{n}-1\right)\left(\frac{p}{n}-2\right)}{2\,.\,3}x^3+\text{ \&c.}$$

$$=1\pm\frac{p}{n}x+\frac{p(p-n)}{2\,.\,n^2}x^2\pm\frac{p(p-n)(p-2n)}{2\,.\,3\,.\,n^3}x^3+\text{ \&c.}\qquad(18)$$

If in this formula we make $p=1$, it becomes

$$(1\pm x)^{\frac{1}{n}}=1\pm\frac{1}{n}x+\frac{1-n}{2.n^2}x^2\pm\frac{(1-n)(1-2n)}{2\,.\,3\,.\,n^3}x^3+\text{ \&c.}\qquad(19)$$

$$=1\pm\frac{1}{n}x-\frac{(n-1)}{2\,.\,n^2}x^2\pm\frac{(n-1)(2n-1)}{2\,.\,3\,.\,n^3}x^3-\text{ \&c.}\qquad(20)$$

Formula (18) may be expressed also in the following manner:

$$(1\pm x)^{\frac{p}{n}}=1\pm\frac{p}{n}x+\frac{p(p-n)}{n\,.\,2n}x^2\pm\frac{p(p-n)(p-2n)}{n\,.\,2n\,.\,3n}x^3+$$

$$\frac{p(p-n)(p-2n)(p-3n)}{n\,.\,2n\,.\,3n\,.\,4n}x^4\pm\text{ &c.}\qquad(21)$$

and a similar notation may be employed in (19) and (20).

For example, to find the square root of $1+x$; that is, to expand $(1+x)^{\frac{1}{2}}$ into a series. Making $n=2$ in (20), we find

$$(1+x)^{\frac{1}{2}}=1+\frac{1}{2}x-\frac{1}{2\,.\,2^2}x^2+\frac{3}{2\,.\,3\,.\,2^3}x^3-\frac{3\,.\,5}{2\,.\,3\,.\,4\,.\,2^4}x^4+\text{ &c.}$$

or reducing the denominators as in (21),

$$(1+x)^{\frac{1}{2}}=1+\frac{1}{2}x-\frac{1}{2.4}x^2+\frac{3}{2.4.6}x^3-\frac{3.5}{2.4.6.8}x^4+$$

$$\frac{3.5.7}{2.4.6.8.10}x^5-\frac{3.5.7.9}{2.4.6.8.10.12}x^6+\text{ &c.}$$

EXAMPLES.

1. $(a+x)^{\frac{1}{3}}=a^{\frac{1}{3}}\left(1+\frac{x}{a}\right)^{\frac{1}{3}}$

$$=a^{\frac{1}{3}}\left(1+\frac{1}{3}\cdot\frac{x}{a}-\frac{2}{3.6}\left(\frac{x}{a}\right)^2+\frac{2\,.\,5}{3\,.\,6\,.\,9}\left(\frac{x}{a}\right)^3-\text{ &c.}\right)$$

2. $(c^2-x^2)^{\frac{3}{4}}=(c^2)^{\frac{3}{4}}\left(1-\frac{x^2}{c^2}\right)^{\frac{3}{4}}$

$$=c^{\frac{3}{2}}\left(1-\frac{3}{2^2}\cdot\frac{x^2}{c^2}-\frac{3}{2^5}\cdot\frac{x^4}{c^4}-\frac{5}{2^7}\cdot\frac{x^6}{c^6}-\text{ &c.}\right)$$

3. $(a\pm b)^{\frac{p}{n}}=a^{\frac{p}{n}}\left(1\pm\frac{b}{a}\right)^{\frac{p}{n}}$

$$=a^{\frac{p}{n}}\left(1\pm\frac{p}{n}\cdot\frac{b}{a}+\frac{p(p-n)}{2\,.\,n^2}\cdot\frac{b^2}{a^2}\pm\frac{p(p-n)(p-2n)}{2\,.\,3\,.\,n^3}\cdot\frac{b^3}{a^3}+\text{ &c.}\right)$$

46. ***Expansion of binomials with negative fractional exponents.***

To obtain a formula for this case we have only to make $\frac{p}{n}$ negative in formula (18) obtained in the last article. This will be effected by making either p or n negative; therefore for n substitute —n, and we have

$$(1\pm x)^{-\frac{p}{n}}=1\mp\frac{p}{n}x+\frac{p(p+n)}{2\,.\,n^2}x^2\mp\frac{p(p+n)(p+2n)}{2\,.\,3\,.\,n^3}x^3+\ \&\text{c} \quad (22)$$

Making n negative in (19), we have

$$(1\pm x)^{-\frac{1}{n}}=1\mp\frac{1}{n}x+\frac{n+1}{2.n^2}x^2\mp\frac{(n+1)(2n+1)}{2\,.\,3\,.\,n^3}x^3+\ \&\text{c.} \quad (23)$$

In these formulæ we may express the denominators as in (21).

For example, to expand $(a-b)^{-\frac{1}{2}}$ or $\frac{1}{\sqrt{a-b}}$, we have

$$(a-b)^{-\frac{1}{2}}=a^{-\frac{1}{2}}\times\left(1-\frac{b}{a}\right)^{-\frac{1}{2}}=\frac{1}{\sqrt{a}}\left(1-\frac{b}{a}\right)^{-\frac{1}{2}}$$

[making $x=\frac{b}{a}$, $n=2$, in (23)]

$$=\frac{1}{\sqrt{a}}\left(1+\frac{1}{2}\cdot\frac{b}{a}+\frac{3}{2.4}\left(\frac{b}{a}\right)^2+\frac{3.5}{2.4.6}\left(\frac{b}{a}\right)^3+\ \&\text{c.}\right)$$

EXAMPLES.

1. $(1-x^2)^{-\frac{2}{3}}=1+\frac{2}{3}x^2+\frac{2.5}{3.6}x^4+\frac{2.5.8}{3.6.9}x^6+$ &c.

2. $(y^2-x^2)^{-\frac{1}{4}}=\frac{1}{\sqrt{y}}\left(1+\frac{1}{4}\cdot\frac{x^2}{y^2}+\frac{5}{2.4^2}\cdot\frac{x^4}{y^4}+\frac{5.9}{2.3.4^3}\cdot\frac{x^6}{y^6}+\ \&\text{c.}\right)$

3. $(a+b)^{-\frac{1}{5}}=\frac{1}{a^{\frac{1}{5}}}\left(1-\frac{1}{5}\cdot\frac{b}{a}+\frac{6}{2.5^2}\cdot\frac{b^2}{a^2}-\frac{6.11}{2.3.5^3}\cdot\frac{b^3}{a^3}+\ \&\text{c.}\right)$

4. $\frac{c^2}{\sqrt{(c^2-x^2)}}=c\left(1+\frac{1}{2}\cdot\frac{x^2}{c^2}+\frac{3}{2.4}\cdot\frac{x^4}{c^4}+\frac{3.5}{2.4.6}\cdot\frac{x^6}{c^6}+\frac{3.5.7}{2.4.6.8}\cdot\frac{x^8}{c^8}+\ \&\text{c.}\right)$

47. *Involution of polynomials.*

For example, to expand $(x+y+z)^3$. In formula (15) make $a=x+y$, $b=z$, $m=3$,

$$(x+y+z)^3=(x+y)^3+3(x+y)^2z+3(x+y)z^2+z^3.$$

Then expanding $(x+y)^3$ &c.

$$(x+y)^3=x^3+3x^2y+3xy^2+y^3,$$
$$3(x+y)^2z=3x^2z+6xyz+3y^2z,$$
$$3(x+y)z^2=3xz^2+3yz^2,$$
$$z^3=z^3;$$

The sum of these equations gives

$$(x+y+z)^3=x^3+3x^2y+3xy^2+y^3+3x^2z+6xyz+3y^2z+3xz^2+3yz^2+z^3.$$

To expand $(a+b+c-d)^4$, we have by our formula

$$(x+y)^4=x^4+4x^3y+6x^2y^2+4xy^3+y^4$$

in which make $x=a+b$, $y=c-d$; we shall find

$$\begin{aligned}(a+b+c-d)^4=a^4&+4a^3b+6a^2b^2+4ab^3+b^4+4a^3c+12a^2bc\\&+12ab^2c+4b^3c-4a^3d-12a^2bd-12ab^2d-4b^3d\\&+6a^2c^2+12abc^2+6b^2c^2-12a^2cd-24abcd\\&-12b^2cd+6a^2d^2+12abd^2+6b^2d^2+4ac^3-12ac^2d\\&+12acd^2-4ad^3+4bc^3-12bc^2d+12bcd^2-4bd^3\\&+c^4-4c^3d+6c^2d^2-4cd^3+d^4.\end{aligned}$$

EXAMPLES.

1. $(a+by+cy^2)^3=a^3+3a^2by+(3a^2c+3ab^2)y^2+(6abc+b^3)y^3$
$+(3ac^2+3b^2c)y^4+3bc^2y^5+c^3y^6.$

2. $(1-x+x^2)^6=1-6x+21x^2-50x^3+90x^4-126x^5+141x^6$
$-126x^7+90x^8-50x^9+21x^{10}-6x^{11}+x^{12}.$

3. $(a^3+a^2b+ab^2+b^3)^4=a^{12}+4a^{11}b+10a^{10}b^2+20a^9b^3+31a^8b^4$
$+40a^7b^5+44a^6b^6+40a^5b^7+31a^4b^8$
$+20a^3b^9+10a^2b^{10}+4ab^{11}+b^{12}.$

EXTRACTION OF ANY ROOT OF A NUMBER.

48. Any root of a number may be found by the binomial theorem by dividing the number into two parts and considering it as a binomial. Thus, to find the 3d root of 28, divide it into 27 and 1; then

$$\sqrt[3]{28}=(27+1)^{\frac{1}{3}}=27^{\frac{1}{3}}\left(1+\frac{1}{27}\right)^{\frac{1}{3}}=3\left(1+\frac{1}{27}\right)^{\frac{1}{3}}.$$

Now to develop $\left(1+\frac{1}{27}\right)^{\frac{1}{3}}$, in (20) Art. 45, make $x=\frac{1}{27}$, $n=3$,

$$\left(1+\frac{1}{27}\right)^{\frac{1}{3}}=1+\frac{1}{3}\cdot\frac{1}{27}-\frac{2}{2.3^2}\cdot\left(\frac{1}{27}\right)^2+\frac{2.5}{2.3.3^3}\cdot\left(\frac{1}{27}\right)^3-\&c.$$

$$=1+\frac{1}{81}-\frac{1}{6561}+\frac{5}{1594323}-\&c.$$

The first three terms of this series are equal to $\frac{6641}{6561}$, or 1.0122 nearly. The fourth term is less than .00001; so that if we wish to find the root to four places of decimals only, this term, as well as all the succeeding terms, may be neglected. Therefore we have

$$\sqrt[3]{28}=3\left(1+\frac{1}{27}\right)^{\frac{1}{3}}=3\times1.0122=3.0366 \text{ nearly.}$$

We divided 28 into two such parts that one of them was an exact power of the third degree, and also the greatest contained in 28. The general method of proceeding is expressed in the following rule: *To find the n^{th} root of any number, first find by trial the nearest integral root (a); divide the given number into two parts, one of which is the n^{th} power of (a); and consider these two parts as the terms of a binomial which may be developed by the binomial theorem.*

49. Sometimes it is expedient to use the power next greater than the given number. Thus, to find the fifth root of 30 we may put

$$\sqrt[5]{30}=(32-2)^{\frac{1}{5}}=32^{\frac{1}{5}}\left(1-\frac{2}{32}\right)^{\frac{1}{5}}=2\left(1-\frac{1}{16}\right)^{\frac{1}{5}}.$$

Then by (20), Art. 45, making $x=\frac{1}{16}$, $n=5$,

$$2\left(1-\frac{1}{16}\right)^{\frac{1}{5}}=2\left[1-\frac{1}{5}\cdot\frac{1}{16}-\frac{4}{2.5^2}\cdot\left(\frac{1}{16}\right)^2-\frac{4.9}{2.3.5^3}\cdot\left(\frac{1}{16}\right)^3-\&c.\right]$$

$$=2\left(1-\frac{1}{80}-\frac{1}{3200}-\frac{3}{256000}-\&c.\right)$$

(taking three terms of the series,)

$$=2\times\frac{3159}{3200}=1.9744 \text{ to within } .0001.$$

The greatest integral power of the 5th degree *less* than 30 is 1, which would lead to a *divergent* series.

50. If the root is desired to a greater number of places, a greater number of terms of the series must be taken. By the following method, however, we may approximate to any degree of accuracy with the use of only the first two terms. Thus, to find the square root of 8, we have

$$\sqrt{8}=(9-1)^{\frac{1}{2}}=3\left(1-\frac{1}{9}\right)^{\frac{1}{2}}$$

$$=3\left(1-\frac{1}{2}\cdot\frac{1}{9}-\frac{1}{2.2^2}\cdot\left(\frac{1}{9}\right)^2-\&c.\right)$$

(taking the first two terms,)

$$=3\times\frac{17}{18}=\frac{17}{6}=2.83 \text{ nearly.}$$

Taking 2.83 as a *first approximation*, we now divide 8 into two parts, one of which is the square of 2.83. We then have

$$\sqrt{8}=[(2.83)^2-0.0089]^{\frac{1}{2}}=2.83\left(1-\frac{89}{(283)^2}\right)^{\frac{1}{2}}$$

$$=2.83\left(1-\frac{1}{2}\cdot\frac{89}{(283)^2}-\&c.\right)$$

$$=2.83-\frac{89\times2.83}{2\times283\times283}=2.83-\frac{89}{56600}$$

$=2.83-0.001572=2.828428$, which is correct within 0.000001 or $\frac{1}{10^6}$.

Taking 2.828428 as a second approximation, we shall find

$$\sqrt{8}=[(2.828428)^2-0.000004951184]^{\frac{1}{2}}.$$

$$=2.828428\left(1-\frac{4951184}{(2828428)^2}\right)^{\frac{1}{2}}$$

$$=2.828428\left(1-\frac{1}{2}\cdot\frac{4951184}{(2828428)^2}\right)$$

$$=2.828428-\frac{4951184}{2\times2828428\times1000000}$$

$$=2.828428-0.000000875254$$

=2.82842,71247,46, which is correct in the last place; that is, within $\frac{1}{10^{12}}$.

It is readily shown that the third term will not affect the 12th place of decimals. This term will be

$$2.828428\times\frac{1}{2\times2^2}\times\frac{(4951184)^2}{(2828428)^4}$$

which may be expressed approximately thus,

$$\frac{3}{10}\times\frac{(5{,}000{,}000)^2}{(8{,}000{,}000{,}000{,}000)^2}=\frac{3\times(5\times10^6)^2}{10\times(8\times10^{12})^2}=\frac{75}{64\times10^{18}}<\frac{1}{10^{12}}.$$

51. The approximation may be effected also in the following manner.*

Suppose the m^{th} root of any number N is sought. Having found an approximate root a, let b be the difference between a^m and N, or

$$N=a^m\pm b$$

Then $N^{\frac{1}{m}}$ will be equal to a, *plus* or *minus* some number which call z; that is, let

$$N^{\frac{1}{m}}=a\pm z, \text{ or } N=(a\pm z)^m.$$

Then we have

$$a^m\pm b=(a\pm z)^m=a^m\pm ma^{m-1}z+\frac{m(m-1)}{2}a^{m-2}z^2\pm \text{ \&c.}$$

or

$$b=ma^{m-1}z\pm\frac{m(m-1)}{2}a^{m-2}z^2+ \text{ \&c.} \qquad (A)$$

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[*] Francœur, Algèbre, n°. 488.

Now for a first approximation neglect the second and the following terms, and take simply

$$b=ma^{m-1}z$$

whence we derive

$$z=\frac{b}{ma^{m-1}}.$$

This value of z, substituted in the second term of (A), gives

$$b=ma^{m-1}z\pm\frac{m(m-1)}{2}a^{m-2}z\times\frac{b}{ma^{m-1}}$$

$$=\left(ma^{m-1}\pm\frac{(m-1)b}{2a}\right)z$$

whence we derive a second value of z,

$$z=\frac{2ab}{2ma^{m}\pm(m-1)b}.$$

In this value substitute the value of $b=\pm(N-a^{m})$ and it becomes

$$z=\frac{\pm2a(N-a^{m})}{2ma^{m}+(m-1)(N-a^{m})}=\frac{\pm2a(N-a^{m})}{(m-1)N+(m+1)a^{m}}.$$

Finally, substituting this value of z, we have

$$N^{\frac{1}{m}}=a\pm z$$

$$=a+\frac{2a(N-a^{m})}{(m-1)N+(m+1)a^{m}}=a\times\frac{(m+1)N+(m-1)a^{m}}{(m-1)N+(m+1)a^{m}}. \quad (24)$$

With this formula we may, by successive substitutions, approximate to any degree of accuracy.

For example, find the square root of 8. In the formula make $m=2$, we have

$$N^{\frac{1}{2}}=a\times\frac{3N+a^{2}}{N+3a^{2}},$$

In this let $N=8$, and substitute as a first approximation, $a=3$. Then we have

$$8^{\frac{1}{2}}=3\times\frac{24+9}{8+27}=3\times\frac{33}{35}=\frac{99}{35}=2.829.$$

Then substituting $a=2.829$, we have

$$8^{\frac{1}{2}}=2.829\times\frac{24+8.003241}{8+24.009723}=2.82842{,}71247{,}5$$

which agrees with the root found in Art. 50.

In the same way we may approximate very rapidly to any root of a number. The general formula will become for the particular cases

$$N^{\frac{1}{2}}=a\times\frac{3N+a^2}{N+3a^2},\quad N^{\frac{1}{3}}=a\times\frac{2N+a^3}{N+2a^3},$$

$$N^{\frac{1}{4}}=a\times\frac{5N+3a^4}{3N+5a^4},\quad N^{\frac{1}{5}}=a\times\frac{3N+2a^5}{2N+3a^5},\ \&c.^*$$

This very convenient method of approximation was first investigated in a general manner by HUTTON. See his "Tracts," Vol. I. p. 45, *et seq.* A similar method, however, had been before employed in the extraction of the cube root by Mr. J. DODSON, and also by Dr. HALLEY.

52. An important facilitation in the ordinary mode of extracting the square root is also suggested by considering the binomial square. Let N represent the number whose square root is sought, and a an approximate root correct to n places of decimals; and let x be the *correction* to be added to a, or the difference between a and the true root; then the first significant figure of x must be in the $(n+1)^{\text{th}}$ place, and $a+x$ represents the true root of N; that is,

$$N=(a+x)^2=a^2+2ax+x^2.$$

Now since the first significant figure of x is in the $(n+1)^{\text{th}}$ place, the first significant figure of x^2 is at least in the $(2n+1)^{\text{th}}$ place. Hence, if we desire the root to $2n$ places only we may neglect x^2, and the above expression becomes

$$N=a^2+2ax,$$

---

* A first approximation may be found to five places of figures with the aid of logarithms, (for which see Chapter VI.,) and by the above methods we may then extend it to any required number of decimals.

whence we have

$$x=\frac{N-a^2}{2a}.$$

That is, *having found an approximate root a, subtract its square from the given number, and divide the remainder by 2a for the remaining figures of the root, which will be accurate to twice as many places as there are in a.*

To illustrate this, take the above example, viz: to find the square root of 8.

```
             8 | 2.82842,71247,
             4 |
           48 | 400
              | 384
          562 | 1600
              | 1124
         5648 | 47600
              | 45184
        56564 | 241600
              | 226256
       565682 | 1534400
              | 1131364
  5,6,5,6,8,4 | 403036 |,71247,
              | 395979 |
                  7057
                  5657
                  1400
                  1131
                   269
                   226
                    43
                    39
                     4
```

Here $N=8$

$a=2.82842$

$N-a^2=.00004,03036$

$2a=5.65684$

$x=.00000,71247$

$a+x=2.82842,71247$ (true in the last place.)

CHAPTER V.

EXPONENTIAL EQUATIONS.

53. ***An exponential equation is one in which the unknown quantity is an exponent.*** Thus,

$$a^x=b$$

is an exponential equation from which, a and b being known, the value of x may be found.

It is evident that if b is some exact power or root of a, the value of x is the exponent of this power or root. For example, if $a=2$, $b=8$, the equation becomes

$$2^x=8,$$

in which x is evidently $=3$, since $2^3=8$.

If $a=16$, $b=2$, then $16^x=2$; from which we find $x=\frac{1}{4}$, since $16^{\frac{1}{4}}=2$, or $\sqrt[4]{16}=2$.

If $a=2$, $b=\frac{1}{64}$, then $2^x=\frac{1}{64}$; from which we find $x=-6$, for $2^{-6}=\frac{1}{2^6}=\frac{1}{64}$, (Art. 20.)

If $a=\frac{1}{1000}$, $b=10$, then $\left(\frac{1}{1000}\right)^x=10$; from which we find $x=-\frac{1}{3}$, for $\left(\frac{1}{1000}\right)^{-\frac{1}{3}}=(1000)^{\frac{1}{3}}=10$.

54. When b is not an exact power or root of a, the solution of the equation is not so readily effected. There are methods, however, of approximating to the value of x as nearly as we please. The following is that of *Lagrange.* Take the equation

$$10^x=500.$$

Now $10^2=100$ and $10^3=1000$; that is, 10^2 is too small and 10^3 is too great. Therefore the value of x is between 2 and 3, and is equal to 2+ some fraction. Represent this fractional part of x by $\frac{1}{x'}$; that is, let

$$x=2+\frac{1}{x'},$$

then our equation becomes

$$10^{2+\frac{1}{x'}}=500,$$

or (Arts. 5 and 21,)

$$10^2\times 10^{\frac{1}{x'}}=500,$$

whence we have

$$10^{\frac{1}{x'}}=\frac{500}{10^2}=5.$$

Involving to the power denoted by x', this becomes

$$10=5^{x'},$$

which is another exponential equation, from which we are to find the value of x'. Now $5^1=5<10$, and $5^2=25>10$. Therefore x' is between 1 and 2. Let

$$x'=1+\frac{1}{x''},$$

then we have in the same manner as above

$$5^{1+\frac{1}{x''}}=10, \text{ or } 5\times 5^{\frac{1}{x''}}=10, \text{ or } 5^{\frac{1}{x''}}=\frac{10}{5}=2, \text{ or}$$

$$5=2^{x''},$$

from which we are to find x''. We have $2^2=4<5$, $2^3=8>5$; therefore x is between 2 and 3. Let

$$x''=2+\frac{1}{x'''},$$

then

$$2^{2+\frac{1}{x'''}}=5, \text{ or } 2^{\frac{1}{x'''}}=\frac{5}{2^2}=\frac{5}{4}, \text{ or}$$

$$\left(\frac{5}{4}\right)^{x'''}=2,$$

in which we find

$$\left(\frac{5}{4}\right)^3=\frac{125}{64}<\frac{128}{64} \text{ or } 2,$$

and

$$\left(\frac{5}{4}\right)^4=\frac{625}{256}>\frac{512}{256} \text{ or } 2;$$

therefore x''' is between 3 and 4.

This process may be continued at pleasure, but we may now consider 3 as the approximate value of x'''. By substitution we shall find

$$x''=2+\frac{1}{x'''}=2+\frac{1}{3}=\frac{7}{3}$$

$$x'=1+\frac{1}{x''}=1+\frac{3}{7}=\frac{10}{7}$$

$$x=2+\frac{1}{x'}=2+\frac{7}{10}=\frac{27}{10}.$$

Therefore $\frac{27}{10}$ is an approximate value of x; that is, $10^{\frac{27}{10}}$, or the 10th root of the 27th power of 10, is nearly equal to 500.

The exponent $\frac{27}{10}$ may also be expressed in the decimal form 2.7 and we may write

$$10^{2.7}=500.$$

55. As another example take the equation

$$8^x=20,$$

in which x is evidently between 1 and 2.

Let $x=1+\frac{1}{x'}$, then we have

$$8^{1+\frac{1}{x'}}=20, \text{ or } 8^{\frac{1}{x'}}=\frac{20}{8}=\frac{5}{2}, \text{ or}$$

$$\left(\frac{5}{2}\right)^{x'}=8,$$

in which x is between 2 and 3.

Let $x'=2+\frac{1}{x''}$, then

$$\left(\frac{5}{2}\right)^{2+\frac{1}{x''}}=8, \text{ or } \left(\frac{5}{2}\right)^{\frac{1}{x''}}=8\times\left(\frac{2}{5}\right)^2=\frac{32}{25}, \text{ or}$$

$$\left(\frac{32}{25}\right)^{x''}=\frac{5}{2},$$

in which x'' is between 3 and 4.

Let $x''=3+\frac{1}{x'''}$, then

$$\left(\frac{32}{25}\right)^{3+\frac{1}{x'''}}=\frac{5}{2}, \text{ or } \left(\frac{32}{25}\right)^{\frac{1}{x'''}}=\frac{5}{2}\times\left(\frac{25}{32}\right)^{3}=\frac{78125}{65536}, \text{ or}$$

$$\left(\frac{78125}{65536}\right)^{x'''}=\frac{32}{25},$$

which equation will be more convenient if we employ decimals. Taking the decimals to two places, it is approximately

$$(1.19)^{x'''}=1.28$$

in which x''' is found to be between 1 and 2.

Let $x'''=1+\frac{1}{x^{\text{iv}}}$, then

$$(1.19)^{1+\frac{1}{x^{\text{iv}}}}=1.28, \text{ or } (1.19)^{\frac{1}{x^{\text{iv}}}}=\frac{1.28}{1.19}=1.08, \text{ or}$$

$$(1.08)^{x^{\text{iv}}}=1.19,$$

in which x^{iv} is found to be between 2 and 3. Taking 2 as its approximate value, we shall have

$$x'''=1+\frac{1}{x^{\text{iv}}}=1+\frac{1}{2}=\frac{3}{2}$$

$$x''=3+\frac{1}{x'''}=3+\frac{2}{3}=\frac{11}{3}$$

$$x'=2+\frac{1}{x''}=2+\frac{3}{11}=\frac{25}{11}$$

$$x=1+\frac{1}{x'}=1+\frac{11}{25}=\frac{36}{25}=1.44.$$

The approximate value of x is therefore $\frac{36}{25}$, or 1.44; whence

$$8^{\frac{36}{25}}=20, \text{ or } 8^{1.44}=20 \text{ nearly.}$$

That is, "the 25th root of the 36th power of 8 is nearly 20." As to the decimal form of the exponent 1.44, we may remark that it is equal to $1\frac{44}{100}$ or $\frac{144}{100}$, and may therefore be read as "the 100th root of the 144th power." So also in the equation

$$10^{2.778}=600$$

the exponent is equivalent to $\frac{2778}{1000}$, and the equation may be read, "the 1000th root of the 2778th power of 10 is equal to 600." If the decimal were taken to four places, we should supply the denominator 10000 or 10^4; if to five places, the denominator 10^5; and in general, *a decimal fraction with n places of decimals, may be converted into a vulgar fraction by placing under it* 10^n *and removing the decimal point.*

The decimal form is the most convenient when the exponent is not an exact fraction.

56. If in the equation $a^x=b$ one of the quantities (a and b) is less than unity, and the other greater, the value of x will be *negative.* Thus, in the equation

$$2^x=\frac{1}{8}$$

we find $x=-3$, for $2^{-3}=\frac{1}{2^3}=\frac{1}{8}$.

In the equation $\left(\frac{2}{3}\right)^x=3.375$, we find $x=-3$, for $\left(\frac{2}{3}\right)^{-3}=\left(\frac{3}{2}\right)^3=\frac{27}{8}=3.375$;

In the equation $8^x=\frac{1}{2}$ we find $x=-\frac{1}{3}$, for $8^{-\frac{1}{3}}=\frac{1}{\sqrt[3]{8}}=\frac{1}{2}$.

An equation of this kind is solved as follows. Take, for example,

$$8^x=\frac{1}{20}.$$

Then since x is negative, let $x=-y$, and we have

$$8^{-y}=\frac{1}{20} \text{ or } \frac{1}{8^y}=\frac{1}{20},$$

whence $$8^y=20,$$

which may be solved as in the preceding article. We shall find $y=1.44$, and $x=-y=-1.44$. Therefore

$$8^{-1.44}=\frac{1}{20}.$$

EXAMPLES.

1. Find the value of x in the equation $3^x=15$. Ans. $x=2.46$.

2 Find the value of x in the equation $10^x=3$. Ans. $x=0.477$.

3. Find the value of x in the equation $10^x=300$.
Ans. $x=2.477$.

4. Find the value of x in the equation $10^x=\frac{1}{3}$.
Ans. $x=-0.477$.

CHAPTER VI.

NATURE AND USE OF LOGARITHMS.

57. The nature of logarithms will be readily understood by considering the exponential equation

$$a^x=b.$$

We have seen in the last chapter that if a and b are given, the value of x may be determined. In fact, a and b may be any two numbers whatever, and x will have a corresponding value which will satisfy the equation. If we suppose a to have a *constant* value as 10, and b to be successively 1, 2, 3, &c., we shall have the equations

$$10^x=1,\ 10^x=2,\ 10^x=3,\ \&c.,$$

from which the value of x in each case may be found by the methods already given, or more expeditiously by those given in Chapter VII. We have in the first equation $x=0$, for $10^0=1$, (Art. 26,) and in the others

$$x=0.301 \text{ or } 10^{0.301}=2 \text{ or } 10^{\frac{301}{1000}}=2$$

$$x=0.477 \quad 10^{0.477}=3 \quad 10^{\frac{477}{1000}}=3$$

$$x=0.602 \quad 10^{0.602}=4 \quad 10^{\frac{602}{1000}}=4$$

&c. &c. &c.

And in the equations

$$10^{x}=11,\ 10^{x}=12, \text{ \&c.}$$

we shall find

$$x=1.041 \text{ or } 10^{1.041}=11$$

$$x=1.079 \quad 10^{1.079}=12$$

&c. &c.

Again, making $b=\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, &c., and determining x from the equations

$$10^{x}=\frac{1}{2},\ 10^{x}=\frac{1}{3},\ 10^{x}=\frac{1}{4}, \text{ \&c.}$$

by Art. 56, we shall find

$$x=-0.301 \text{ or } 10^{-0.301}=\frac{1}{2} \text{ or } \frac{1}{10^{0.301}}=\frac{1}{2}$$

$$x=-0.477 \quad 10^{-0.477}=\frac{1}{3} \quad \frac{1}{10^{0.477}}=\frac{1}{3}$$

&c. &c. &c.

We see then that ***any number whatever may be considered as some power*** [*] ***of the number* 10. *For all numbers greater than unity the exponent of* 10 *will be positive, and for all numbers less than unity the exponent of* 10 *will be negative.***

58. In the same manner any other number (except unity) may be made to express all numbers whatever by affixing to it a proper

---

[*] The word *power* is frequently applied, as in the text, to all quantities affected with exponents. In this sense, $a^{\frac{1}{n}}$ is a power of a whose degree is $\frac{1}{n}$, $a^{\frac{m}{n}}$ a power whose degree is $\frac{m}{n}$. This generalization of the word power becomes necessary by the extension of the notation of integral powers to negative and fractional exponents.

exponent. It is evident that unity is excepted, since all the powers of 1 are 1.

A constant number being thus assumed, the series of exponents which must be assigned to this number in order to express the series of natural numbers, 1, 2, 3, &c., may be calculated and arranged in tables for use. Such a series of exponents constitutes a *system* of *logarithms*, and the constant number assumed is called the *base* of the system. Hence the following definition:

The logarithm of a number, in any system, is the exponent of the power to which the base of the system must be involved in order to produce that number.

59. The logarithm of a number is denoted by the abbreviation *log.*, or simply by the letter *l.* or *L.*; and since a logarithm is nothing else than an exponent, we often use the notation

$$a^{\log. b} = b,$$

a representing the base of a system, b any number, and log. b the power to which a must be raised in order to produce b.

PROPERTIES OF LOGARITHMS IN GENERAL.

60. ***The logarithm of the product of two or more numbers is equal to the sum of the logarithms of those numbers.***

For let b, c, d, &c. be any numbers, and a the base of any system of logarithms, then we have by the definition of logarithms

$$a^{\log. b} = b$$
$$a^{\log. c} = c$$
$$a^{\log. d} = d$$
$$\&c.$$

Multiply these equations together. The first members being powers of the same quantity are multiplied by adding their exponents (Art. 5,) therefore we have

$$a^{\log. b + \log. c + \log. d + \ldots} = bcd \ldots$$

But we also have, by the definition of logarithms,

$$a^{\log. bcd \ldots} = bcd \ldots$$

therefore we have the equation

$$a^{\log. bcd \ldots}=a^{\log. b+\log. c+\log. d+\ldots}$$

whence $\log. bcd\ldots=\log. b+\log. c+\log. d+\ldots$

If, therefore, it is required to *multiply* two or more numbers together, we have only to take their logarithms from a table and *add* them together; the sum will be found in the table to be the logarithm of the required product.

61. *The logarithm of the quotient of two numbers is equal to the difference of their logarithms.*

For b and c being any two numbers, we have

$$a^{\log. b}=b$$
$$a^{\log. c}=c.$$

Dividing the first equation by the second, we have by Art. 6

$$a^{\log. b-\log. c}=\frac{b}{c}.$$

But we also have $a^{\log.\frac{b}{c}}=\frac{b}{c},$

whence the equation $a^{\log.\frac{b}{c}}=a^{\log. b-\log. c},$

Therefore $\log.\frac{b}{c}=\log. b-\log. c.$

To *divide* one number by another, therefore, we take their logarithms from the table and *subtract* the log. of the divisor from the log. of the dividend; the remainder is found in the table to be the logarithm of the required quotient.

62. *The logarithm of any power of a number is equal to the logarithm of that number multiplied by the exponent of the power.*

For b being any number, we have

$$a^{\log. b}=b.$$

Involve this equation to any power, as the n^{th}. The first member is involved by multiplying the exponent log. b by n, according to Art. 8, and we have

$$a^{n\log. b}=b^{n}.$$

But $a^{\log. b^n}=b^n$,

whence the equation $a^{\log. b^n}=a^{n \log. b}$,

from which we have $\log. b^n=n \log. b$.

Therefore, to involve a given number to any power, we multiply the logarithm of the number by the exponent of the power; the product is the logarithm of the required power.

63. *The logarithm of any root of a number is equal to the logarithm of that number divided by the index of the root.*

For to extract the n^{th} root of the equation

$$a^{\log. b}=b,$$

we divide the exponents of each member by n, according to Art. 9; we thus find

$$a^{\frac{\log. b}{n}}=b^{\frac{1}{n}}=\sqrt[n]{b}.$$

We also have, by the definition of logarithms,

$$a^{\log. \sqrt[n]{b}}=\sqrt[n]{b}.$$

Therefore $a^{\log. \sqrt[n]{b}}=a^{\frac{\log. b}{n}}$,

whence $\log. \sqrt[n]{b}=\dfrac{\log. b}{n}$.

Therefore, to extract any root of a number, we divide the logarithm of the number by the index of the root; the quotient is the logarithm of the required root.

64. *The logarithm of unity is zero in all systems.*

For a being the base of a system, we have by Art. 26, whatever is the value of a

$$a^0=1.$$

That is, $\log. 1=0$.

This follows also from Art. 60; for

$$\log. b\times 1=\log. b+\log. 1,$$

and $b\times 1$ is b; so that this equation is simply

$$\log. b=\log. b+\log. 1,$$

or $0=\log. 1$.

65. ***The logarithm of the reciprocal of a number is the logarithm of that number taken negatively.***

For b being any number, we have

$$a^{\log. b}=b,$$

whence we have

$$\frac{1}{a^{\log. b}}=\frac{1}{b},$$

or by Art. 20

$$a^{-\log b}=\frac{1}{b},$$

$a^{lg\,\frac{1}{b}}=\frac{1}{b}.$

that is,

$$\log. \frac{1}{b}=-\log. b.$$

This also follows from Arts. 61 and 64, thus:

$$\log. \frac{1}{b}=\log. 1-\log. b=0-\log. b=-\log. b.$$

66. ***The logarithm of the base of a system is unity.***

For $a^1=a$;

that is, $\log. a=1.$

67. The student should make himself familiar with the mode of expressing algebraical quantities in logarithms. The following examples will be understood by referring to the preceding principles.

1. $\log. \left(\frac{abc}{de}\right)=\log. a+\log. b+\log. c-(\log. d+\log. e).$

2. $\log. (a^m b^n c^p \ldots)=m\log. a+n\log. b+p\log. c\ldots$

3. $\log. \left(\frac{a^m b^{\overset{n}{m}}}{c^p}\right)=m\log. a+n\log. b-p\log. c.$

4. $\log. \sqrt{ab}=\frac{1}{2}(\log. a+\log. b).$

5. $\log. \sqrt[3]{\frac{a^2bc}{d^2}}=\frac{1}{3}(2\log. a+\log. b+\log. c-2\log. d).$

6. $\log. (a^3\sqrt[4]{a^3})=\log. \left(a^3a^{\frac{3}{4}}\right)=3\log. a+\frac{3}{4}\log. a=\frac{15}{4}\log. a.$

7. $\log. \sqrt{a^2-x^2}=\frac{1}{2}[\log. (a+x)+\log. (a-x)].$

8. log. $\sqrt{a^2+x^2}$. Here to apply logarithms we must resolve a^2+x^2 into factors. Let $2ax=y^2$; then we have $a^2+x^2=a^2+2ax+x^2-2ax=(a+x)^2-y^2=(a+x+y)(a+x-y)$; $\log.\sqrt{a^2+x^2}=\frac{1}{2}[\log.(a+x+y)+\log.(a+x-y)]$.

The value of y is found by logarithms from $y^2=2ax$,

$$2\log.y=\log.2+\log.a+\log.x,$$

$$\log.y=\frac{1}{2}(\log.2+\log.a+\log.x).$$

9. $\log.\frac{1}{(a+x)^3}=-\log.(a+x)^3=-3\log.(a+x)$.

10. $\log.\left(\frac{\log.a}{\log.b}\right)=\log.\log.a-\log.\log.b$.

11. $\log.(x^2-1)-\log.(x+1)+\log.(x-1)=\log.(x-1)^2$.

12. $\log.5+\log.7-\log.9-\log.11+\log.x=\log.\frac{35}{99}x$.

13. $3\log.x+\log.y-\log.(x^2+xy)-\log.(y^2+xy)=\log.\left(\frac{x}{x+y}\right)^2$ *.

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* In some treatises, logarithms are defined to be "a series of numbers in arithmetical progression corresponding to a series of numbers in geometrical progression." But this is not a definition, but a consequence following from the definition; it is moreover calculated to confuse the student, who finds in his tables that the natural numbers are arranged in arithmetical progression, and not the logarithms.

If x is the logarithm of b in the system whose base is a, we have

$$a^x=b,\ a^{2x}=b^2,\ a^{3x}=b^3,\ a^{4x}=b^4,\ \&c.,$$

whence we have the logarithms x, $2x$, $3x$, $4x$, &c., in arith. prog. corresponding to the numbers b, b^2, b^3, b^4, &c., in geom. prog.

Again, if we multiply the equation $a^x=b$ by the equation $a^y=c$, we shall obtain

$$a^x=b,\ a^{x+y}=bc,\ a^{x+2y}=bc^2,\ a^{x+3y}=bc^3,\ \&c.,$$

whence we have logs. x, $x+y$, $x+2y$, $x+3y$, &c., in arith. prog. and the numbers b, bc, bc^2, bc^3, &c., in geom. prog.

COMMON LOGARITMS.

68. In the system of logarithms in common use, the base is 10; so that to construct a table of these, we have to solve the equations

$$10^x=1,\quad 10^x=2,\quad 10^x=3,\ \&c.$$

We have at once the logarithms of the powers of 10, thus:

$$\begin{aligned}
&\log.\ 1=0, \text{ (Art. 64.)}\\
&\log.\ 10=1, \text{ (Art. 66.)}\\
&\log.\ 100=\log.\ 10^2=2\log.\ 10=2, \text{ (Art. 62.)}\\
&\log.\ 1000=\log.\ 10^3=3\log.\ 10=3,\\
&\log.\ 10000=\log.\ 10^4=4\log.\ 10=4,\\
&\quad\&c. \qquad\qquad \&c.
\end{aligned}$$

also

$$\begin{aligned}
&\log.\ 0.1=\log.\frac{1}{10}=-\log.\ 10=-1, \text{ (Art. 65.)}\\
&\log.\ 0.01=\log.\frac{1}{100}=-\log.\ 10^2=-2,\\
&\log.\ 0.001=\log.\frac{1}{1000}=-\log.\ 10^3=-3,\\
&\log.\ 0.0001=\log.\frac{1}{10000}=-\log.\ 10^4=-4,\\
&\quad\&c. \qquad\qquad \&c.
\end{aligned}$$

That is, the logarithm of a number composed of the figure 1 and ciphers, is equal to the number of places by which the figure 1 is removed from the units place, and is positive when this figure is to the left, and negative when it is to the right of the unit's place.

69. It follows from this, that if a number is

between 1 and 10, its log. is between 0 and 1,
" 10 " 100, " 1 " 2,
" 100 " 1000, " 2 " 3,
" 1000 " 10000, " 3 " 4,
&c. &c.

and if between 0.1 and 1, its log. is between —1 and 0,
" 0.01 " 0.1, " —2 " —1,
" 0.001 " 0.01, " —3 " —2,
" 0.0001 " 0.001, " —4 " —3,
&c. &c.

70. *The integral part of a logarithm is called its* characteristic, *and sometimes its* index.

From the preceding table we perceive that if a number is between 1 and 10, that is if it contains only one integral figure, its logarithm is between 0 and 1 and consists of the characteristic 0 and a decimal fraction. If a number is between 10 and 100, it contains two integral figures, and its logarithm consists of the characteristic 1 and a decimal fraction; and, generally, a number between 10^n and 10^{n+1} contains n +1 integral figures, and its logarithm has the characteristic n.

Again, if a number is between 0.1 and 1, its logarithm is between —1 and 0, and is negative; thus, log. 0.3456 = —0.46143. But this mode of expressing negative logarithms is never employed; it is found more convenient to regard the logarithm of a decimal fraction as composed of two parts, a negative characteristic and a positive decimal fraction. According to this method we shall have log. 0.3456 = —1 + .53857, which is equivalent to —0.46143. In like manner we shall have log. 0.03456 = —2 + .53857; and, generally, a number being between $\frac{1}{10^n}$ and $\frac{1}{10^{n-1}}$, (that is, between 10^{-n} and 10^{-n+1},) its logarithm will have the characteristic $-n$. For the sake of conciseness, the characteristic is placed next the decimal with the minus sign over it; thus, log. 0.3456 = $\bar{1}$.53857, log. 0.03456 = $\bar{2}$.53857, which must not be mistaken for —1.53857, —2.53857.

From these considerations we derive the following rule:

The characteristic of the logarithm of a number is equal to the number of places by which the first significant figure of that number is removed from the unit's place, and is positive when this figure is to the left, negative when it is to the right, and zero when it is in the unit's place.

71. Let b represent any number whose first significant figure is in the unit's place; its logarithm will consist of the characteristic 0 and a decimal fraction, which represent by d. Then we shall have

$$\log.\ b = 0 + d,$$

$$\log.\ 10\,b = \log.\ 10 + \log.\ b = 1 + d,$$

$$\log.\ 100\,b = \log.\ 100 + \log.\ b = 2 + d,$$

and in general, $\log.\ 10^n b = \log.\ 10^n + \log.\ b = n + d,$

also, $\log.\ \frac{b}{10} = -\log.\ 10 + \log.\ b = -1 + d,$

$$\log.\ \frac{b}{100} = -\log.\ 100 + \log.\ b = -2 + d,$$

and in general, $\log.\ \frac{b}{10^n} = -\log.\ 10^n + \log.\ b = -n + d.$

These equations prove, in a general manner, the preceding rule for the characteristic. We see also that the decimal part (d) of the logarithm is the same for all the numbers b, $10b$, $10^n b$, $\frac{b}{10}$, $\frac{b}{10^n}$. But multiplying or dividing a number by 10 does not change its significant figures, but merely removes the decimal point to the right or to the left. Hence we have another important property of common logarithms, that ***the same decimal part is common to the logarithms of all numbers composed of the same significant figures.***

All this is exemplified in the following table:

$$\begin{array}{lrl}
\log. & 12345 & = 4.09149 \\
\log. & 1234.5 & = 3.09149 \\
\log. & 123.45 & = 2.09149 \\
\log. & 12.345 & = 1.09149 \\
\log. & 1.2345 & = 0.09149 \\
\log. & 0.12345 & = \bar{1}.09149 \\
\log. & 0.012345 & = \bar{2}.09149 \\
\log. & 0.0012345 & = \bar{3}.09149 \\
\log. & 0.00012345 & = \bar{4}.09149 \\
 & \&c. & \quad \&c.
\end{array}$$

72. On these properties depends the usual arrangement of the tables of common logarithms. None but whole numbers are placed in the table, and the characteristics of their logarithms are omitted. To find the logarithm of any number from the tables, we neglect the decimal point and take out the logarithm corresponding to the *figures* of which our number consists; we then prefix the charac-

teristic according to the position of the decimal point, by the rule given above. On the other hand, in finding the number corresponding to a given logarithm, we regard only the decimal part of the logarithm and take out the corresponding *figures* from the column of numbers; the characteristic then determines the unit's place and the position of the decimal point.*

73. As the use of negative characteristics is apt to produce confusion in calculation by the mixture of positive and negative quantities, it is most generally avoided by adding 10 to the characteristic, which is thus rendered positive. The logarithm may then be used in this form throughout a calculation, at the end of which we must make the proper allowance, adding or subtracting 10, as the case may be. No error can arise from this, as it is hardly possible in a calculation to make a mistake of 10,000,000,000.

Thus, the log. of 0.3456 becomes $-1+.53857+10=9.53857$, and the log. of 0.03456 becomes $-2+.53857+10=8.53857$. We have then this rule:

The characteristic of the logarithm of a decimal fraction is equal to 10 minus *the number of places by which the first significant figure is removed from the unit's place.*

If the negative characteristic were greater than 10, we should render it positive by adding 100. Thus, instead of $\overline{19}.23471$ we should use 81.23471, and reject 100 at the end of the calculation.

74. *To find the product of two numbers.*

This is effected by Art. 60; that is, by *adding* the logarithms.

EXAMPLE.

To find the product of 4125 and 312. The process will be as follows:

log. 4125 = 3.61542
log. 312 = 2.49415
log. 1287000 = 6.10957

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* Particular directions for using tables are not here given, because every collection of tables is accompanied with the necessary explanations.

We find in the tables that the decimal part of the logarithm 6.10957 corresponds to the *figures* 1287; but the characteristic 6 shows that the first significant figure is removed 6 places from the unit's place. We must therefore add three ciphers, which gives 1287000 for the true product.

75. The use of logarithms is particularly convenient when the continued product of several numbers is sought.

EXAMPLE.

To find the continued product of 41.87, 2.385, 9.4, 0.012 and 0.85.

$$\begin{array}{ll}
\text{log. } 41.87 & = 1.62190 \\
\text{log. } 2.385 & = 0.37749 \\
\text{log. } 9.4 & = 0.97313 \\
\text{log. } 0.012 & = 8.07918 \text{ (Art 73.)} \\
\text{log. } 0.85 & = 9.92942 \\
\hline
\text{log. } 9.5746 & = 0.98112
\end{array}$$

Here, since the characteristics of log 0.012 and log. 0.85 are each 10 too great, that of the sum of the logarithms is 20 too great. Rejecting 20, then, the log. is 0.98112, which is the log. of 9.5746 the required product.

76. *To find the quotient of two numbers.*

This is effected by Art. 61; that is, by *subtracting* the log. of the divisor from the log. of the dividend.

EXAMPLES.

To find the quotient of 5979.3 divided by 432.18.

$$\begin{array}{ll}
\text{log. } 5979.3 & = 3.77665 \\
\text{log. } 432.18 & = 2.63566 \\
\hline
\text{log. } 13.835 & = 1.14099
\end{array}$$

The required quotient is therefore 13.835.

To find the quotient of 597.93 divided by 43218.

$$\begin{array}{rl}
10+\text{log. } 597.93 & = 12.77665 \\
\text{log. } 43218 & = 4.63566 \\
\hline
\text{log. } 0.013835 & = 8.14099
\end{array}$$

Here, the characteristic of log. 43218 being greater than that of log. 597.93, the remainder would be negative, to avoid which we add 10 to the log. of 597.93. The remainder is then the logarithm of a decimal fraction, and the position of the decimal point is determined by the rule in Art. 73.

77. As it is often more convenient to add than to subtract, the subtraction of logarithms is changed to addition by using the *arithmetical complement.*

The arithmetical complement of a logarithm is the remainder after subtracting it from 10.

The most convenient mode of finding it is to subtract the first significant figure on the right hand from 10, and all the others from 9.

Thus, the ar. co. of 5.83116 is 4.16884
" of 2.56467 is 7.43533
" of 1.13450 is 8.86550

The arithmetical complement of the logarithm of a decimal fraction will be found by subtracting it from 20.

Thus, log. 0.08792 = 8.94409 and ar. co. = 11.05591
log. 0.00057 = 6.75587 " = 13.24413

In performing division by logarithms, instead of subtracting the log. of the divisor we may add its arithmetical complement and reject 10 *from the sum.*

For if b and c are any two numbers

$$\text{log. } \frac{b}{c} = \text{log. } b - \text{log. } c = \text{log. } b + 10 - \text{log. } c - 10 =$$
$$\text{log. } b + (\text{ar. co.}) \text{ log. } c - 10.$$

EXAMPLES.

To divide 1976.3 by 42.896.

log. 1976.3 = 3.29585
(ar. co.) log. 42.896 = 8.36758

log. 46.071 = 1.66343

The required quotient is therefore 46.071.

To divide 19.763 by 0.0042896.

$$\begin{array}{rlrl} & \log. & 19.763 = & 1.29585 \\ (\text{ar. co.}) & \log. & 0.0042896 = & 12.36758 \\ \hline & \log. & 4607.1 = & 3.66343 \end{array}$$

The required quotient is therefore 4607.1.

78. *To involve a number to any power.*

This is effected by Art. 62; that is, by ***multiplying*** the log. of the number by the exponent of the power.

EXAMPLES.

To find the fourth power of 5.7

$$\begin{array}{lrr} \log. & 5.7 = & 0.75587 \\ & & 4 \\ \hline \log. & 1055.6 = & 3.02348 \end{array}$$

Therefore $(5.7)^4 = 1055.6$.

To find the third power of 0.12

$$\begin{array}{lrr} \log. & 0.12 = & 9.07918 \\ & & 3 \\ \hline \log. & 0.001728 = & 7.23754 \end{array}$$

Therefore $(0.12)^3 = 0.001728$.

Here log. 0.12 is 10 too great, and since we multiply by 3 the product must be 30 too great. But we reject only 20, in order to avoid a negative characteristic; it is therefore 10 too great, and is the logarithm of a decimal fraction.

79. *To extract any root of a number.*

This is effected by Art. 63; that is, by ***dividing*** the log. of the number by the index of the root.

EXAMPLES.

To find the square root of 87825

$$\begin{array}{lrr} \frac{1}{2}\log. & 87825 = \frac{1}{2} \,| & 4.94362 \\ \hline \log. & 296.35 = & 2.47181 \end{array}$$

Therefore $\sqrt{87825} = 296.35$.

If the logarithm has a negative characteristic, add as many times 10 *as there are units in the index of the required root; the quotient will exceed its true value by* 10.

To find the third root of 0.57894.

$$\begin{array}{r}\tfrac{1}{3}\,|\,30 + \log.\ 0.57894 = \tfrac{1}{3}\,|\,29.76263 \\ \hline \log.\ 0.83345 = \quad 9.92088\end{array}$$

Therefore $\sqrt[3]{0.57894} = 0.83345$.

If we consider 9.76263 as the log. of 0.57894, we have to add 20 only, since it is already 10 too great.

80. *To find the fourth term of a proportion.*

The fourth term of a proportion is found by multiplying the second and third terms together, and dividing by the first. In logarithms this is effected by adding the logarithms of the second and third terms and subtracting the logarithm of the first term. Or instead of subtracting the log. of the first term, we may add its arithmetical complement. (Art. 77.)

EXAMPLES.

To find the fourth term of the proportion 33.16 : 398 :: 4.893 : x,

$$\begin{array}{rrl}(\text{ar. co.})\ \log. & 33.16 = & 8.47939 \\ \log. & 398 = & 2.59988 \\ \log. & 4.893 = & 0.68958 \\ \hline \log. & 58.729 = & 1.76885\end{array}$$

Therefore $x = 58.729$.

To find the fourth term of the proportion

$$0.0059768 : 478.3 :: 0.0076821 : x,$$

$$\begin{array}{rrl}(\text{ar. co.})\ \log. & 0.0059768 = & 12.22353 \\ \log. & 478.3 = & 2.67970 \\ \log. & 0.0076821 = & 7.88548 \\ \hline \log. & 614.77 = & 2.78871\end{array}$$

Therefore $x = 614.77$.

81. Complicated quantities of various kinds are readily computed by logarithms.

EXAMPLE.

To find the value of the fraction

$$\frac{\sqrt[3]{957}\times(410)^2\times(0.00379)^{\frac{1}{2}}}{(4.653)^{\frac{1}{5}}\times(2.39)^4}$$

$\frac{1}{3}$ log. 957	$=\frac{1}{3}$ | 2.98091		$=$	0.99364
2 log. 410	$=2$ | 2.61278		$=$	5.22556
$\frac{1}{2}$ log. 0.00379	$=\frac{1}{2}$ | 17.57864		$=$	8.78932
$\frac{1}{5}$ log. 4.653	$=\frac{1}{5}$ | 0.66773	$=0.13355$	$=$ (ar. co.)	9.86645
4 log. 2.39	$=4$ | 0.37840	$=1.51360$	$=$ (ar. co.)	8.48640
		log. 2298.1	$=$	3.36137

Therefore the value of the fraction is 2298.1.

82. The following examples will afford practice in all the usual operations by logarithms.

1. $\dfrac{(0.335)^4\times\sqrt{18.6}\times\sqrt[4]{5}}{897.65\times(0.00017)^{\frac{1}{2}}}=0.0069398.$

2. $\dfrac{(25)^{\frac{1}{3}}\times(26)^{\frac{1}{4}}\times(27)^{\frac{1}{5}}}{(28)^{\frac{1}{6}}\times(29)^{\frac{1}{7}}\times(30)^{\frac{1}{8}}}=2.9597.$

3. $\dfrac{(0.00015)^{\frac{1}{2}}\times(0.0000156)^5}{(587.69)^3}=\dfrac{5.5747}{(10)^{35}}.$

4. $\sqrt{(149.16)^2-(17.8)^2}=148.09.$ (See Ex. 7, Art. 67.)

5. $\dfrac{(567.8)^2-(467.2)^2}{(15)^7\times0.53}=0.0011498.$

6. $\sqrt{(49.755)^2+(37.845)^2}=62.513.$ (See Ex. 8, Art. 67.)

83. *To solve an exponential equation by logarithms.*

In the exponential equation

$$a^x=b,$$

if $a=10$, the value of x is found, by mere ***inspection***, from a table of common logarithms, since in this case x is the common logarithm of b. But if a has any other value, as 8, then the value of x is the log. of b in the system whose base is 8, and may be found as follows. Take the equation

$$8^x=50.$$

Taking the logarithms of both members, we have, by Art. 62,

$$x \log. 8=\log. 50,$$

whence we have $x=\dfrac{\log. 50}{\log. 8}=\dfrac{1.69897}{0.90309}=1.8813.$

The division of 1.69897 by 0.90309 may also be performed by logarithms.

$$\begin{array}{l}
\log. 1.69897=0.23019\\
\underline{\log. 0.90309=9.95573}\\
\log. \ \ 1.8813=0.27446
\end{array}$$

84. The complete solution of an exponential equation is therefore as follows:

Taking the logarithms of the equation

$$a^x=b,$$

we have $$x \log. a=\log. b,$$

whence $$x=\frac{\log. b}{\log. a},$$

and again taking the logarithms

$$\log. x=\log. \frac{\log. b}{\log. a}=\log. \log. b-\log. \log. a.$$

EXAMPLE

To find the value of x in the equation

$$12^x=100.$$

$$\begin{array}{ll}
\log. 100=2.00000, & \log. 2.00000=0.30103\\
\log. \ \ 12=1.07918, & \underline{\log. 1.07918=0.03309}\\
 & \log. \ \ 1.8533=0.26794
\end{array}$$

Therefore $x=1.8533$, or $12^{1.8533}=100$; that is, 1.8533 is the logarithm of 100 in the system whose base is 12.

CHAPTER VII.

CONSTRUCTION OF LOGARITHMS.

85. The great practical utility of logarithms has induced mathematicians to expend much labor in the construction and correction of tables, in which are given the logarithms of all numbers from 1 to 10,000, to 100,000, and in some to 1,000,000; and this not only to 5 places of decimals, but to 7, 10, and even 14 places. Partial tables have been constructed to 20 places, and the logarithms of all the prime numbers below 1100 have been calculated to 61 places. This extreme accuracy is of course unnecessary in practice. Indeed seven places are considered sufficient for nearly every practical purpose in astronomy and navigation.*

86. We have already seen that the construction of logarithms requires the solution of the equation

$$a^x = b,$$

in which a is the base of a system, b any number, and x the logarithm of that number. The value of x can be found *exactly* only in a few cases. When the base of the system is 10, for example, the logarithms of the integral powers of 10 are 1, 2, 3, 4, &c., but the logarithms of all other numbers are *incommensurable;* that is, they cannot be expressed exactly by any fraction or by any number of decimals.* We may, however, approximate to their values with any required degree of accuracy. The most convenient method of approximation is by series, which we proceed to explain.

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* LALANDE, in the preface to his pocket volume of logarithmic tables, (edition of 1805,) makes the following declaration : "I have calculated some hundreds of eclipses, and I have rarely used any other tables than those which I now publish ; for my experience proves, that observations are very rarely so accurate as to require any others." Great accuracy, however, is now attained in observation, and seven places are very generally employed in astronomical calculations. In nautical calculations more than five places are seldom if ever used.

We shall make use of the following principle.

87. ***Any finite number whatever may be represented by a fraction whose numerator and denominator are both indefinitely small.***

Thus, the number 3 may be represented by

$$\frac{3}{1}, \text{ or by } \frac{\frac{3}{100}}{\frac{1}{100}}, \frac{\frac{3}{10000}}{\frac{1}{10000}}, \text{ \&c., or by } \frac{.0003}{.0001}, \frac{.0000003}{.0000001}, \text{ \&c.}$$

in short, by any fraction whose numerator is three times its denominator; consequently, by a fraction whose numerator and denominator are indefinitely small, provided their ratio or quotient is 3.

In general, x being any number, we may represent it by $\frac{\frac{x}{N}}{\frac{1}{N}}$, in which $\frac{x}{N}$ is x times $\frac{1}{N}$ whatever value is assigned to N. If N is indefinitely great, $\frac{x}{N}$ and $\frac{1}{N}$ are indefinitely small, but their quotient is still x. We conclude, then, that if h and k are two indefinitely small quantities whose ratio is x, we may always represent x by putting

$$x=\frac{h}{k}.\dagger$$

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* A rigid demonstration of this may be found in Bourdon, Algébre, p. 342. Paris, 1837.

† The algebraic symbol expressing an indefinitely small quantity is 0, and the symbol expressing an indefinitely great quantity is ∞. The quotient obtained by dividing any finite number by ∞ is 0; and the quotient obtained by dividing any finite number by 0 is ∞. The reasoning in the text might therefore be given thus: x being any finite number, we may put $x=\frac{\frac{x}{\infty}}{\frac{1}{\infty}}=\frac{0}{0}$. The symbol $\frac{0}{0}$ is therefore an indeterminate expression, which may represent any number whatever. See *Davies' Bourdon*, p. 101, *et seq.*

TO FIND THE LOGARITHM OF A GIVEN NUMBER.

88. The method of solving the exponential equation $a^x=b$, given in Chapter V, does not afford a general expression for the value of x in terms of a and b, and is moreover too laborious to be of any use in the construction of tables of logarithms. The general solution is as follows, where we first convert a and b into binomials in order to apply the binomial theorem.

Let $a=1+p$, $b=1+n$, and $x=\frac{h}{k}$, h and k being indefinitely small, according to Art. 87. Then our equation becomes

$$(1+p)^{\frac{h}{k}}=1+n.$$

Involving this to the power k, we have, by Art. 8,

$$(1+p)^h=(1+n)^k. \qquad (25)$$

Expanding both members by formula (10) of Art. 33, we have

$$1+hp+\frac{h(h-1)}{2}p^2+\frac{h(h-1)(h-2)}{2\,.\,3}p^3+\text{ \&c.}=1+kn$$
$$+\frac{k(k-1)}{2}n^2+\frac{k(k-1)(k-2)}{2\,.\,3}n^3+\text{ \&c.} \qquad (26)$$

Canceling the 1 from both members, and dividing by k, we have

$$\frac{h}{k}p+\frac{h}{k}\cdot\frac{h-1}{2}p^2+\frac{h}{k}\cdot\frac{(h-1)(h-2)}{2\,.\,3}p^3+\text{ \&c.}=n+\frac{k-1}{2}n^2$$
$$+\frac{(k-1)(k-2)}{2\,.\,3}n^3+\text{ \&c.} \qquad (27)$$

Substitute for $\frac{h}{k}$ its value x; we shall find

$$x\left(p+\frac{h-1}{2}p^2+\frac{(h-1)(h-2)}{2\,.\,3}p^3+\text{ \&c.}\right)=n+\frac{k-1}{2}n^2$$
$$+\frac{(k-1)(k-2)}{2\,.\,3}n^3+\text{ \&c.} \qquad (28)$$

Now $h-1$ differs from -1 only by the indefinitely small quantity h; we may therefore use -1 instead of $h-1$, without any appreciable error. For the same reason we may omit h from all the factors $h—2$, $h—3$, &c., and k from all the factors $k—1$, $k—2$, $k—3$, &c. Equation (28) then becomes

$$x\left(p+\frac{-1}{2}p^2+\frac{(-1)(-2)}{2\,.\,3}p^3+\text{ \&c.}\right)=n+\frac{-1}{2}n^2+\frac{(-1)(-2)}{2\,.\,3}n^3+\text{ \&c.}$$

or $$x\left(p—\frac{1}{2}p^2+\frac{1}{3}p^3—\text{ \&c.}\right)=n—\frac{1}{2}n^2+\frac{1}{3}n^3—\text{ \&c.} \quad (29)$$

From this equation we now find the value of x by dividing by the compound factor $p—\frac{1}{2}p^2+\frac{1}{3}p^3—$ &c., which gives

$$x=\frac{n—\frac{1}{2}n^2+\frac{1}{3}n^3—\frac{1}{4}n^4+\frac{1}{5}n^5—\text{ \&c.}}{p—\frac{1}{2}p^2+\frac{1}{3}p^3—\frac{1}{4}p^4+\frac{1}{5}n^5—\text{ \&c.}} \quad (30)$$

in which the second member is a known quantity, since n and p are known, when a and b are given, from the equations

$$a=1+p,\ b=1+n;\ \text{whence } p=a—1,\ n=b—1.$$

These values substituted in (30) give

$$x=\log.b=\frac{(b—1)—\frac{1}{2}(b—1)^2+\frac{1}{3}(b—1)^3—\frac{1}{4}(b—1)^4+\text{ \&c.}}{(a—1)—\frac{1}{2}(a—1)^2+\frac{1}{3}(a—1)^3—\frac{1}{4}(a—1)^4+\text{ \&c.}} \quad (31)$$

which is the general solution of the equation $a^x=b$, and gives the value of the logarithm of any number b in the system whose base is a.

89. In using this formula to calculate a table of logarithms, we first find the value of the denominator, which is constant, since it involves only the base of the system. Represent this denominator by A, and its reciprocal by M; that is, let

$$A=(a—1)—\frac{1}{2}(a—1)^2+\frac{1}{3}(a—1)^3—\frac{1}{4}(a—1)^4+\text{ \&c.} \quad (32)$$

and $$M=\frac{1}{A}=\frac{1}{(a—1)—\frac{1}{2}(a—1)^2+\frac{1}{3}(a—1)^3—\frac{1}{4}(a—1)^4+\text{ \&c.}} \quad (33)$$

Formula (31) will then become

$$\log.b=M[b—1—\frac{1}{2}(b—1)^2+\frac{1}{3}(b—1)^3—\frac{1}{4}(b—1)^4+\text{ \&c.}] \quad (34)$$

This constant factor M, whose value depends upon the base of the system, is called the MODULUS.

TO FIND THE NUMBER CORRESPONDING TO A GIVEN LOGARITHM.

90. This is the inverse of the preceding problem, and the solution of it must give the number in terms of its logarithm and of the base of the system (or of the modulus, which depends upon the base.) If the given logarithm were integral, the number to which it corresponded might be found by involving the base to the power indicated by the logarithm; but in all other cases evolution would also be necessary, as a single example will show. The number whose logarithm is .3 in the system whose base is a, is $a^{.3}$ or $a^{\frac{3}{10}}$, or $\sqrt[10]{a^3}$; hence the number will be found by involving the base to the third power and then extracting the tenth root. In the following solution, however, we obtain an expression for the required number, in which the given logarithm does not enter as an exponent.

91. Resume the fundamental equation

$$a^x=b, \text{ or } b=a^x,$$

and as before, let $a=1+p$, $x=\frac{h}{k}$; (h and k being indefinitely small.) Then we have

$$b=(1+p)^{\frac{h}{k}}, \text{ whence } b^k=(1+p)^h.$$

Now, by the binomial theorem, we have

$$(1+p)^h=1+hp+\frac{h(h-1)}{2}p^2+\frac{h(h-1)(h-2)}{2\,.\,3}p^3+\text{ \&c.} \qquad (35)$$

Neglecting h in all the factors $h-1$, $h-2$, &c. this equation becomes, as in Art. 88,

$$\begin{aligned}(1+p)^h&=1+h(p-\tfrac{1}{2}p^2+\tfrac{1}{3}p^3-\text{ \&c.})\\ &=1+h[a-1-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3-\text{ \&c.}]\\ &=1+hA, \text{ (Art. 89,)} \qquad (36)\end{aligned}$$

that is, $$b^k=1+hA,$$

whence $$b=(1+hA)^{\frac{1}{k}}. \qquad (37)$$

Expanding the second member by formula (19) of Art. 45, by making $n=k$, $x=hA$, we have

$$b=1+\frac{1}{k}(hA)+\frac{(1-k)}{2\,.\,k^2}(hA)^2+\frac{(1-k)(1-2k)}{2\,.\,3\,.\,k^3}(hA)^3+\text{\&c.} \quad (38)$$

Neglecting k, $2k$, &c. in the factors $1-k$, $1-2k$, &c. this equation becomes

$$b=1+\frac{h}{k}A+\frac{1}{2}\cdot\frac{h^2}{k^2}\cdot A^2+\frac{1}{2.3}\cdot\frac{h^3}{k^3}\cdot A^3+\text{\&c.} \qquad (39)$$

and restoring $x=\frac{h}{k}$,

$$b=1+xA+\frac{1}{2}x^2A^2+\frac{1}{2.3}x^3A^3+\text{\&c.} \qquad (40)$$

which is the required formula, expressing any number b in terms of its logarithm x and of A the reciprocal of the modulus of the system.

If we substitute for x and A their values

$$x=\log.b, \quad A=\frac{1}{M},$$

formula (40) becomes

$$b=1+\frac{\log.b}{M}+\frac{1}{2}\left(\frac{\log.b}{M}\right)^2+\frac{1}{2.3}\left(\frac{\log.b}{M}\right)^3+\frac{1}{2.3.4}\left(\frac{\log.b}{M}\right)^4$$

$$+\frac{1}{2.3.4.5}\left(\frac{\log.b}{M}\right)^5+\frac{1}{2.3.4.5.6}\left(\frac{\log.b}{M}\right)^6+\text{\&c.} \qquad (41)$$

TO DETERMINE THE MODULUS FROM THE BASE.

92. Although (33) gives the value of M in terms of a, yet the series in the denominator is not convergent when a is greater than 2, as will be seen by substituting 3, 4, &c. for a. It is rendered convergent by the following simple transformation, given by LAGRANGE.

The base of the system being a, the logarithm of any root of a, as the m^{th} root, is $\frac{1}{m}$, by the definition of logarithms, (Art. 58.)

That is, $$\log.\,a^{\frac{1}{m}}=\frac{1}{m}.$$

But since (34) expresses the value of the logarithm of any number, we may employ it to express that of the logarithm of $a^{\frac{1}{m}}$ by substituting $b=a^{\frac{1}{m}}$. We thus obtain

$$\log.\ a^{\frac{1}{m}}=\frac{1}{m}=M\left[a^{\frac{1}{m}}-1-\frac{1}{2}\left(a^{\frac{1}{m}}-1\right)^2+\frac{1}{3}\left(a^{\frac{1}{m}}-1\right)^3-\&c.\right]$$

whence we derive

$$M=\frac{1}{m\left[a^{\frac{1}{m}}-1-\frac{1}{2}\left(a^{\frac{1}{m}}-1\right)^2+\frac{1}{3}\left(a^{\frac{1}{m}}-1\right)^3-\&c.\right]}\qquad(42)$$

or $$A=m\left[a^{\frac{1}{m}}-1-\frac{1}{2}\left(a^{\frac{1}{m}}-1\right)^2+\frac{1}{3}\left(a^{\frac{1}{m}}-1\right)^3-\&c.\right]\qquad(43)$$

In this formula the series may always be rendered convergent. For m being indefinite, we may assume it so great that $a^{\frac{1}{m}}$ shall be less than 2; then $a^{\frac{1}{m}}-1$ will be less than 1, and its powers will consequently decrease. We see also that the greater m is assumed the more rapidly will the series converge.

93. Let us apply this formula to finding the modulus of the common system in which $a=10$. Assume $m=32$; then $a^{\frac{1}{m}}=10^{\frac{1}{32}}$.

The 32nd root of 10 may be found by five successive extractions of the square root, since $32=2^5$. We shall find*

$$\sqrt{10}=3.16227{,}76601{,}7$$
$$\sqrt[4]{10}=1.77827{,}94100{,}4$$
$$\sqrt[8]{10}=1.33352{,}14321{,}6$$
$$\sqrt[16]{10}=1.15478{,}19846{,}9$$
$$\sqrt[32]{10}=1.07460{,}78283{,}2$$

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* This operation will be shortened by employing the methods explained in Chapter IV, Arts. 50, 51, 52.

Therefore $a^{\frac{1}{m}}-1=.07460,78283,2$, and the calculation of the series will be as follows, representing $a^{\frac{1}{m}}-1$ by z.

$$\begin{array}{rlrr}
z = & .074607828\ 32 & & \\
-\frac{1}{2}z^2 = & & - & .002783163\ 99 \\
+\frac{1}{3}z^3 = & 138430\ 55 & & \\
-\frac{1}{4}z^4 = & & - & 7746\ 07 \\
+\frac{1}{5}z^5 = & 462\ 33 & & \\
-\frac{1}{6}z^6 = & & - & 28\ 74 \\
+\frac{1}{7}z^7 = & 1\ 84 & & \\
-\frac{1}{8}z^8 = & & - & 12 \\
+\frac{1}{9}z^9 = & 1 & & \\
\hline
 & .074746723\ 05 & - & .002790938\ 92 \\
 & .002790938\ 92 & & \\
\hline
 & .071955784\ 13 & & \\
 & 32 & & \\
\hline
\mathcal{A} = & 2.302585092\ 16 & & \\
M = & \frac{1}{\mathcal{A}} = .434294482 & &
\end{array}$$

The value of M to sixty-one places is .43429,44819,03251,82765, 11289,18916,60508,22943,97005,80366,65661,14453,8.

94. It is necessary in calculations like the above, to employ throughout the process at least two more decimals than are retained in the result, so that the error occasioned by the neglected figures may not affect the last figure retained. We here employ eleven places of decimals in the calculation, but retain only nine in the result, which may therefore be relied upon as correct in the last place.

The above is the direct method of finding the modulus from the base, but it may also be obtained with the aid of naperian logarithms. (See Art. 99.)

TO DETERMINE THE BASE FROM THE MODULUS.

95. This is effected at once by (41), for if $b=$ the base of the system $=a$, then log. $b=1$, and this formula becomes

$$a=1+\frac{1}{M}+\frac{1}{2}\left(\frac{1}{M}\right)^2+\frac{1}{2.3}\left(\frac{1}{M}\right)^3+\frac{1}{2.3.4}\left(\frac{1}{M}\right)^4+\text{ \&c.} \qquad (44)$$

In constructing a system of logarithms, we may either assume the base and then determine the modulus by Art. 92; or we may assume the modulus and then determine the base by (44). In the common system, or Briggs', the *base* is assumed $=10$; in the naperian system the *modulus* is assumed $=1$.

NAPERIAN LOGARITHMS.

96. In the system of Napier, the inventor of logarithms, the modulus is assumed $=1$. If therefore we make $M=1$, in (44,) the series will give the value of Napier's base, which is generally represented by e. We have then

$$e=1+1+\frac{1}{2}+\frac{1}{2.3}+\frac{1}{2.3.4}+\frac{1}{2.3.4.5}+\text{ \&c.} \qquad (45)$$

The exponents of the powers of this base are naperian logarithms. They are also called *hyperbolic* logarithms, from the use made of them in the quadrature of the equilateral hyperbola. Although the common system has superseded the use of tables of naperian logarithms in nearly every practical operation, yet the latter possess some important advantages in theoretical investigations, and particularly in the Integral Calculus; advantages resulting principally from the simplicity of the logarithmic formula, from which the factor M is made to disappear by being assumed $=1$.

The value of e is found approximately by taking a number of terms of the series. The calculation is made very simple by remarking that if any term be divided by the number indicating its place in the series, the result is the next succeeding term; thus, the first term divided by 1 gives the second term 1; this divided by 2 gives the third term $\frac{1}{2}$, &c. The calculation will therefore be as follows, where for convenience we designate the successive terms by A, B, C, &c.

$$\begin{array}{lcrl}
1+1 & = & 2. & \\
\frac{1}{2} & = & .5 & (A) \\
\frac{1}{3}A & = & .16666,66666,666\ 67 & (B) \\
\frac{1}{4}B & = & 4166,66666,666\ 67 & (C) \\
\frac{1}{5}C & = & 833,33333,333\ 33 & (D) \\
\frac{1}{6}D & = & 138,88888,888\ 89 & (E) \\
\frac{1}{7}E & = & 19,84126,984\ 13 & (F) \\
\frac{1}{8}F & = & 2,48015,873\ 02 & (G) \\
\frac{1}{9}G & = & 27557,319\ 22 & (H) \\
\frac{1}{10}H & = & 2755,731\ 92 & (I) \\
\frac{1}{11}I & = & 250,521\ 08 & (K) \\
\frac{1}{12}K & = & 20,876\ 76 & (L) \\
\frac{1}{13}L & = & 1,605\ 90 & (M) \\
\frac{1}{14}M & = & 114\ 71 & (N) \\
\frac{1}{15}N & = & 7\ 65 & (O) \\
\frac{1}{16}O & = & 48 & (P) \\
\frac{1}{17}P & = & 3 & \\
\hline
\end{array}$$

$$e = \text{Napier's base} = 2.71828,18284,590\ 46$$

The value of e to sixty-one places is 2.71828,18284,59045, 23536,02874,71352,66249,77572,46928,08355,51550,58417,2.

97. If x represents the logarithm of b in the system whose modulus is M, and x' the logarithm of b in the system whose modulus is M', we have by (34)

$$x = M\,(b-1-\tfrac{1}{2}(b-1)^2+\tfrac{1}{3}(b-1)^3-\text{\&c.}) \qquad (46)$$

$$x' = M'(b-1-\tfrac{1}{2}(b-1)^2+\tfrac{1}{3}(b-1)^3-\text{\&c.}) \qquad (47)$$

Dividing (46) by (47), we have

$$\frac{x}{x'} = \frac{M}{M'},$$

or

$$x : x' :: M : M'. \qquad (48)$$

That is, *the logarithms of the same number in different systems are to each other as the moduli of those systems.*

98. If in (48) $x=\log. b$ in any system, $x'=L.\,b$,* in the naperian system, and $M'=$ modulus of naperian system $=1$, we have

$$\log. b : L.\,b :: M : 1,$$

whence $$\log. b=M\times L.\,b. \qquad (49)$$

That is, *the logarithm of a number in any system is equal to the modulus of that system multiplied by the naperian logarithm of the number.*

99. If $b=a=$ base of the system whose modulus is M, then $\log. b=1$, and (49) becomes

$$1=M\times L.\,a,$$

whence $$M=\frac{1}{L.\,a}. \qquad (50)$$

That is, *the modulus of any system is the reciprocal of the naperian logarithm of the base of the system.*

If in (50) $a=10$, we shall have the value of the modulus of the common system, or

$$M=\frac{1}{L.\,10}.$$

The naperian logarithm of 10, as will be shown hereafter, (Art. 113,) is 2.30258,50930. Therefore

$$\text{Briggs' modulus }=\frac{1}{2.30258{,}50930}=.43429{,}44819,$$

which agrees with the value found by another method in Art. 93.

100. If we substitute in (49) the value of M given by (50), we have

$$\log. b=\frac{L.\,b}{L.\,a}. \qquad (51)$$

That is, *the logarithm of a number to any given base is equal to the naperian logarithm of that number divided by the naperian logarithm of the given base.*

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* We here designate naperian logarithms by the letter $L.$, to distinguish them from other logarithms.

101. If we make $b=e$ in (49), we have

$$\log. e = M \times L.\ e = M. \tag{52}$$

That is, *the logarithm of Napier's base in any system is equal to the modulus of that system.*

Hence, *Briggs' modulus is the common logarithm of Napier's base*, or

$$\log. 2.718281828\ \&c. = .4342944819\ \&c.$$

102. With the aid of these principles, we can construct as many systems of logarithms as we please after having calculated a table of naperian logarithms; for, by (49), we have only to multiply all the logs. of this table by the modulus of the new system, in order to obtain the logarithms in this system; or, by (51), we have only to divide each naperian logarithm by the naperian logarithm of the new base, to obtain the logarithms to that base.

TRANSFORMATIONS OF THE LOGARITHMIC FORMULA.

103. We may now suppose the modulus to be known, since it may be determined for any base by formula (42). We are not, however, yet prepared to construct a table, for (34) is not convergent unless b is less than 2. Various transformations have been proposed, by which it is rendered very convergent and expeditious in practice. The most useful of these we proceed to explain.

104. Since (34) will converge when $b<2$, let $b=1+\frac{d}{n}=\frac{n+d}{n}$, whence $b-1=\frac{d}{n}$, and $\log. b = \log. (n+d) - \log. n$, (Art. 61.) The formula becomes

$$\log.(n+d)-\log. n=M\left(\frac{d}{n}-\frac{d^2}{2n^2}+\frac{d^3}{3n^3}-\frac{d^4}{4n^4}+\frac{d^5}{5n^5}-\&c.\right) \tag{53}$$

Again, let $b=1-\frac{d}{n}=\frac{n-d}{n}$, whence $b-1=-\frac{d}{n}$, and $\log. b = \log. (n-d) - \log. n$. The formula becomes

$$\log.(n-d)-\log. n=M\left(-\frac{d}{n}-\frac{d^2}{2n^2}-\frac{d^3}{3n^3}-\frac{d^4}{4n^4}-\frac{d^5}{5n^5}-\&c.\right) \tag{54}$$

Add (54) to (53); we have

$$\log.(n+d)-2\log.n+\log.(n-d)=M\left(-\frac{d^2}{n^2}-\frac{d^4}{2n^4}-\frac{d^6}{3n^6}-\&c.\right) \quad (55)$$

or transposing and reducing,

$$\log.(n+d)=2\log.n-\log.(n-d)-M\left(\frac{d^2}{n^2}+\frac{d^4}{2n^4}+\frac{d^6}{3n^6}+\ \&c.\right) \quad (56)$$

Subtract (54) from (53); we have

$$\log.\ (n+d)-\ \log.\ (n-d)=M\left(\frac{2d}{n}+\frac{2d^3}{3n^3}+\frac{2d^5}{5n^5}+\ \&c.\right) \quad (57)$$

or transposing and reducing,

$$\log.\ (n+d)=\log.\ (n-d)+2M\left(\frac{d}{n}+\frac{d^3}{3n^3}+\frac{d^5}{5n^5}+\ \&c.\right) \quad (58)$$

These formulæ will always converge when $\frac{d}{n}$ is a proper fraction; that is, when $d<n$. In using (53) log. n must be known before log. $(n+d)$ can be determined. From (54) we find log. n when log. $(n-d)$ is given; from (56) we find log. $(n+d)$ when log. n and log. $(n-d)$ are given; and from (58) we find log. $(n+d)$ when log. $(n-d)$ is given.

105. In (58) let $n+d=p$, $n-d=1$, whence $2n=p+1$, $2d=p-1$, and $\frac{d}{n}=\frac{p-1}{p+1}$; then we have (since log. $(n-d)=$ log. $1=0$)

$$\log.\ p=2M\left[\frac{p-1}{p+1}+\frac{1}{3}\left(\frac{p-1}{p+1}\right)^3+\frac{1}{5}\left(\frac{p-1}{p+1}\right)^5+\ \&c.\right] \quad (59)$$

This formula requires no logarithms to be given, but converges slowly unless p is a small number.

106. In formulæ (53, 54, 56, 58) let $d=1$; then we have

$$\log.(n+1)=\log.n+M\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\frac{1}{4n^4}+\frac{1}{5n^5}-\ \&c.\right) (60)$$

$$\log.(n-1)=\log.n-M\left(\frac{1}{n}+\frac{1}{2n^2}+\frac{1}{3n^3}+\frac{1}{4n^4}+\frac{1}{5n^5}+\ \&c.\right) (61)$$

$$\log.(n+1)=2\log.n-\log.(n-1)-M\left(\frac{1}{n^2}+\frac{1}{2n^4}+\frac{1}{3n^6}+\&c.\right)\quad(62)$$

$$\log.(n+1)=\log.(n-1)+2M\left(\frac{1}{n}+\frac{1}{3n^3}+\frac{1}{5n^5}+\&c.\right)\quad(63)$$

In (58) substitute $n+d=p+1$, $n-d=p$, whence we have $2n=2p+1$, $2d=1$, and $\frac{d}{n}=\frac{1}{2p+1}$; the formula becomes

$$\log.(p+1)=\log.p+2M\left(\frac{1}{2p+1}+\frac{1}{3(2p+1)^3}+\frac{1}{5(2p+1)^5}+\&c.\right)\quad(64)$$

These formulæ will be found convenient in calculating the logarithm of a number, when that of a number greater or less by unity is given.

107. The above would suffice for the computation of a table, but other formulæ have been found, in which the series are more rapidly convergent. That of BORDA may be thus derived. In (58) let

$$n+d=(p-1)^2(p+2)=p^3-3p+2$$
$$n-d=(p+1)^2(p-2)=p^3-3p-2$$

whence we find $n=p^3-3p$, $d=2$, and

$$\log.(n+d)=2\log.(p-1)+\log.(p+2)$$
$$\log.(n-d)=2\log.(p+1)+\log.(p-2)$$

These values substituted in (58) give

$$\log.(p+2)-2\log.(p+1)+2\log.(p-1)-\log.(p-2)=$$
$$2M\left[\frac{2}{p^3-3p}+\frac{1}{3}\left(\frac{2}{p^3-3p}\right)^3+\frac{1}{5}\left(\frac{2}{p^3-3p}\right)^5+\&c.\right]\quad(65)$$

By this formula we may obtain $\log.(p+2)$ when the logarithms of $p+1$, $p-1$ and $p-2$ are given. The series is so convergent that when $p=50$ the first significant figure of the second term is in the 15th place of decimals.

108. A still more convergent formula, but requiring more logarithms to be known, is that of HAROS. In (58) let

$$n+d=(p-3)(p+3)(p-4)(p+4)=p^4-25p^2+144$$
$$n-d=p^2(p-5)(p+5) \qquad\qquad =p^4-25p^2$$

whence we have $n=p^4-25p^2+72$, $d=72$, and the formula becomes

$$\log.(p+5)-\log.(p+4)-\log.(p+3)+2\log.p-\log.(p-3)-\log.(p-4)+\log.(p-5)=$$
$$-2M\left[\frac{72}{p^4-25p^2+72}+\frac{1}{3}\left(\frac{72}{p^4-25p^2+72}\right)^3+\text{\&c.}\right] \quad (66)$$

This formula expresses the relation between seven logarithms, and the series is very convergent.

109. In (64) let $p+1=q^2$, $p=q^2-1=(q-1)(q+1)$; the formula becomes

$$2\log.q=\log.(q-1)+\log.(q+1)+$$
$$2M\left(\frac{1}{2q^2-1}+\frac{1}{3(2q^2-1)^3}+\text{\&c.}\right)$$

from which we derive

$$\log.q=\tfrac{1}{2}\left[\log.(q-1)+\log.(q+1)\right]+$$
$$M\left(\frac{1}{2q^2-1}+\frac{1}{3(2q^2-1)^3}+\text{\&c.}\right) \quad (67)$$

which is a very convenient formula, requiring but two logarithms to be given.

110. An unlimited number of formulæ may be derived from (53) and (54) by a peculiar combination, the law of which will be perceived in the following examples.

In (53) let $d=3, 2, 1$, and in (54) let $d=1, 2, 3$, successively; we shall thus obtain the six following formulæ:

$$\log.(n+3)-\log.n= M\left(\frac{3}{n}-\frac{9}{2n^2}+\frac{27}{3n^3}-\frac{81}{4n^4}+\frac{243}{5n^5}-\&c.\right)(A)$$

$$\log.(n+2)-\log.n= M\left(\frac{2}{n}-\frac{4}{2n^2}+\frac{8}{3n^3}-\frac{16}{4n^4}+\frac{32}{5n^5}-\&c.\right)(B)$$

$$\log.(n+1)-\log.n= M\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\frac{1}{4n^4}+\frac{1}{5n^5}-\&c.\right)(C)$$

$$\log.(n-1)-\log.n=M\left(-\frac{1}{n}-\frac{1}{2n^2}-\frac{1}{3n^3}-\frac{1}{4n^4}-\frac{1}{5n^5}-\&c.\right)(D)$$

$$\log.(n-2)-\log.n=M\left(-\frac{2}{n}-\frac{4}{2n^2}-\frac{8}{3n^3}-\frac{16}{4n^4}-\frac{32}{5n^5}-\&c.\right)(E)$$

$$\log.(n-3)-\log.n=M\left(-\frac{3}{n}-\frac{9}{2n^2}-\frac{27}{3n^3}-\frac{81}{4n^4}-\frac{243}{5n^5}-\&c.\right)(F)$$

Combine these equations thus: $B—3C—D$; we shall find

$$\log.(n+2)-3\log.(n+1)+3\log.n-\log.(n-1)= M\left(\frac{2}{n^3}-\frac{3}{n^4}+\frac{6}{n^5}-\frac{10}{n^6}+\&c.\right) \qquad (68)$$

Make the combination $B—4C—4D\overset{+}{—}E$; then

$$\log.(n+2)-4\log.(n+1)+6\log.n-4\log.(n-1)+\log.(n-2)= -M\left(\frac{6}{n^4}+\frac{20}{n^6}+\frac{63}{n^8}+\frac{204}{n^{10}}+\&c.\right) \qquad (69)$$

Make the combination $A—5B+10C+5D—E$; then

$$\log.(n+3)-5\log.(n+2)+10\log.(n+1)-10\log.n+5\log.(n-1)-\log.(n-2)= M\left(\frac{24}{n^5}-\frac{60}{n^6}+\frac{240}{n^7}-\frac{630}{n^8}+\&c.\right) \qquad (70)$$

Make the combination $A—6B+15C+15D—6E+F$; then

$$\log.(n+3)-6\log.(n+2)+15\log.(n+1)-20\log.n+15\log.(n-1)-6\log.(n-2)+\log.(n-3)= -M\left(\frac{120}{n^6}+\frac{1260}{n^8}+\frac{10584}{n^{10}}+\&c.\right) \qquad (71)$$

Such formulæ may be multiplied at pleasure, as the law of their formation is easily recognized. They express the relation between a number $(m+1)$ of logarithms of consecutive numbers; the terms in the first members are alternately positive and negative; the coefficients are those of the (m^{th}) power of the binomial, and the denominator of the first term of the series is n^m.

Formula (70) is perhaps the most useful, as it is very easy to multiply by the numbers 5 and 10; but the series in (69) and (71) are more convergent than the series in (70).

111. Logarithms may also be found by the following method. Instead of the logarithm of the number itself, we seek that of any root of the number which is less than 2. This logarithm will be found by (34); we then multiply it by the index of the root to obtain the logarithm of the given number, (Arts. 62 and 63).

Thus, in (34) let $b=p^{\frac{1}{n}}$, and we have

$$\log. p^{\frac{1}{n}}=\frac{1}{n}\log. p=M\left[p^{\frac{1}{n}}-1-\frac{1}{2}\left(p^{\frac{1}{n}}-1\right)^2+\frac{1}{3}\left(p^{\frac{1}{n}}-1\right)^3-\&c.\right]$$

$$\log. p=nM\left[p^{\frac{1}{n}}-1-\frac{1}{2}\left(p^{\frac{1}{n}}-1\right)^2+\frac{1}{3}\left(p^{\frac{1}{n}}-1\right)^3-\&c.\right] \qquad (72)$$

In this formula, as n is arbitrary, we may assume it so great that the n^{th} root of the number p will be less than 2; then $p^{\frac{1}{n}}-1$ will be less than 1, and its powers will decrease. This transformation is the same as that given in Art. 92, in finding the modulus; and if we substitute in (72) the value of M, given by (42), we shall have

$$\log. p=\frac{n\left[p^{\frac{1}{n}}-1-\frac{1}{2}\left(p^{\frac{1}{n}}-1\right)^2+\frac{1}{3}\left(p^{\frac{1}{n}}-1\right)^3-\&c.\right]}{m\left[a^{\frac{1}{m}}-1-\frac{1}{2}\left(a^{\frac{1}{m}}-1\right)^2+\frac{1}{3}\left(a^{\frac{1}{m}}-1\right)^3-\&c.\right]} \qquad (73)$$

which is the formula given by **LAGRANGE**. It expresses the logarithm of any number in terms of that number and the base of the system, n and m being arbitrary numbers, but such that $p^{\frac{1}{n}}$ and $a^{\frac{1}{m}}$ are less than 2.

This method is similar to that actually employed by Briggs in the construction of his tables; but the tedious operation of extracting roots which it requires, has been rendered unnecessary by the more expeditious formulæ which we have already given.

APPLICATION OF THE FORMULÆ.

112. In the actual construction of a table, it is necessary to employ the preceding formulæ only in finding the logarithms of ***prime*** numbers; for the logarithms of ***composite*** numbers will be found by adding together the logarithms of their factors, (Art. 60). We shall have to find by the formulæ the logs. of the prime numbers 2, 3, 5, 7, 11, 13, 17, &c., but those of the composite numbers will be derived from these as follows:

$$\begin{array}{ll}
\log.\ 4 = \log.\ 2^2 = 2\log.\ 2, & \log.\ 6 = \log.\ 2 + \log.\ 3, \\
\log.\ 8 = \log.\ 2^3 = 3\log.\ 2, & \log.\ 9 = \log.\ 3^2 = 2\log.\ 3, \\
\log.\ 10 = \log.\ 2 + \log.\ 5, & \log.\ 12 = \log.\ 3 + \log.\ 4, \text{ \&c.}
\end{array}$$

113. If we wish to find naperian logarithms we make $M=1$ in all the formulæ, which then take the simplest form for calculation. But in finding common logarithms we make $M=.4342$, &c., (Art. 93). This value of M may be found directly as in Art. 93, or more conveniently, as in Art. 99, from the naperian logarithm of 10.

The most natural method of finding the naperian logarithm of 10 is to find that of 2, then that of 5, and then that of $10 = L.\ 2 + L.\ 5$. For this purpose, in (64) let $M=1$ and $p=1$; then $L.\,p=0$, and we have

$$L.\ 2 = 2\left(\frac{1}{3} + \frac{1}{3\,.\,3^3} + \frac{1}{5\,.\,3^5} + \frac{1}{7\,.\,3^7} + \text{\&c.}\right)$$

and by actual computation, taking ten terms of the series, we have

$$L.\ 2 = 0.69314,71805,6.$$

Then $L.\,4=2\,L.\,2=1.38629,43611,2$. Making $p=4$ in (64), we have

$$L.\,5=L.\,4+2\left(\frac{1}{9}+\frac{1}{3\,.\,9^3}+\frac{1}{5\,.\,9^5}+\frac{1}{7\,.\,9^7}+\text{\&c.}\right)$$

(taking six terms of the series)

$$=1.38629,43611,2+.22314,25513,1,$$
$$=1.60943,79124,3.$$

Hence $L.\,10=0.69314,71805,6+1.60943,79124,3$,
$$=2.30258,50929,9.$$

From this we find, as in Art. 99, Briggs' modulus = reciprocal of $L.\,10=.43429,44819,03$.

114. The preceding example sufficiently illustrates the mode of using the formulæ. We have only to substitute numbers for letters, and compute the series by adding a number of terms, greater or less according to the degree of accuracy required. It will be observed that, generally, the most convergent formulæ are those which require the greatest number of logarithms to be previously known. Thus, (59), which requires no logarithms to be given, is not so convergent as (64), in which one logarithm is supposed to be known; formulæ (65 .... 71) are still more convergent, requiring three, four, five, or more logarithms to be given.

To find the common logarithms of 2 and 5, we have only to make $M=.434$, &c. in the formulæ and compute the series as above; or we may multiply their naperian logarithms already obtained, by .434, &c., according to Art. 98. We may then proceed to find the logs. of the succeeding prime numbers as follows:

In (64) let $p=2$,

$$\log.\,3=\log.\,2+.828\ \text{\&c.}\left(\frac{1}{5}+\frac{1}{3\,.\,5^3}+\frac{1}{5\,.\,5^5}+\text{\&c.}\right)$$

Let $p=6$, $\log.\,p=\log.\,2+\log.\,3$,

$$\log.\,7=\log.\,2+\log.\,3+.828\ \text{\&c.}\left(\frac{1}{13}+\frac{1}{3\,.\,13^3}+\frac{1}{5\,.\,13^5}+\text{\&c.}\right)$$

Let $p=10$,

$$\log.\,11=\log.\,2+\log.\,5+.828\ \text{\&c.}\left(\frac{1}{21}+\frac{1}{3\,.\,21^3}+\frac{1}{5\,.\,21^5}+\text{\&c.}\right)$$

We might thus compute the entire table with this formula only; but the labour may be much abridged by various devices, some of which we shall here explain.

115. In commencing the construction of a table, we know only the logarithms of the integral powers of the base of the system; from these we may find the logs. of all numbers greater or less than these powers by unity, by the formulæ of Art. 106.

Thus, in Briggs' system we know the logs. of 10, 100, 1000, &c. to be 1, 2, 3, &c. From these we may find those of 9, 11, 99, 101, 999, 1001, &c. by making successively n=10, 100, &c. in (60) and (61); we thus find

$$\log.\ 11=1+.434\ \&c.\left(\frac{1}{10}-\frac{1}{2.100}+\frac{1}{3.1000}-\frac{1}{4.10000}+\&c.\right)$$

$$\log.\ 9=1-.434\ \&c.\left(\frac{1}{10}+\frac{1}{2.100}+\frac{1}{3.1000}+\frac{1}{4.10000}+\&c.\right)$$

$$\log.101=2+.434\ \&c.\left(\frac{1}{10^2}-\frac{1}{2.100^2}+\frac{1}{3.1000^2}-\frac{1}{4.10000^2}+\&c.\right)$$

$$\log.\ 99=2-.434\ \&c.\left(\frac{1}{10^2}+\frac{1}{2.100^2}+\frac{1}{3.1000^2}+\frac{1}{4.10000^2}+\&c.\right)$$

&c. &c.

The similarity of these series enables us to compute them all at the same time, by observing that the significant figures of the first, second, third, &c. terms respectively, are necessarily the same in all the series, but are preceded by a greater or less number of ciphers. We subjoin the computation of the preceding examples, designating the terms by A, B, C, &c.

To find log. 11 and log. 9,

A =	.04342,94481,90	B =	.00217,14724,10
C =	14,47648,27	D =	1,08573,62
E =	8685,89	F =	723,82
G =	62,04	H =	5,43
I =	48	K =	04
	.04357,50878,58		.00218,24027,01

$$\text{log. } 11 = 1 + (.04357{,}50878{,}58 - .00218{,}24027{,}01) = \\ 1.04139{,}26852,$$

$$\text{log. } 9 = 1 - (.04357{,}50878{,}58 + .00218{,}24027{,}01) = \\ 0.95424{,}25094.$$

To find log. 101 and log. 99, we have from the preceding calculation, simply by changing the position of the decimal point,

$$\begin{array}{lr@{\qquad}lr}
A = & .00434{,}29448{,}19 & B = & .00002{,}17147{,}24 \\
C = & 1447{,}65 & D = & 10{,}86 \\
E = & 9 & & \\
\hline
 & .00434{,}30895{,}93 & & .00002{,}17158{,}10
\end{array}$$

$$\text{log. } 101 = 2 + (.00434{,}30895{,}93 - .00002{,}17158{,}10) = \\ 2.00432{,}13738,$$

$$\text{log. } 99 = 2 - (.00434{,}30895{,}93 + .00002{,}17158{,}10) = \\ 1.99563{,}51946.$$

We employ twelve places of decimals in the computation, but retain only ten in the logarithms, which may therefore be relied upon as correct in the last place.

In the same manner we should find

log. 1001 = 3.00043,40775	log. 10001 = 4.00004,34273
log. 999 = 2.99956,54882	log. 9999 = 3.99995,65684

116. These logarithms prepared, we may proceed as follows. We have $1024 = 2^{10}$, and log. 1024 = 10 log. 2; therefore in (53) let $n = 1000$, $d = 24$, and $\frac{d}{n} = .024$; then

$$10 \text{ log. } 2 = \text{log. } 1000 + M[.024 - \tfrac{1}{2}(.024)^2 + \tfrac{1}{3}(.024)^3 - \text{\&c.}]$$

whence, by taking six terms of the series,

$$\text{log. } 2 = \tfrac{1}{10}(3 + .01029{,}99566{,}4) = 0.30102{,}99956{,}64,$$

$$\text{log. } 3 = \text{log. } \sqrt{9} = \tfrac{1}{2} \text{ log. } 9, \quad \text{log. } 5 = \text{log. } 10 - \text{log. } 2.$$

To find log. 7, in (64) let $p=9800=7^2.2.10^2$, $p+1=9801=11^2.3^4$; then we have

$$2 \text{ log. } 11+4 \text{ log. } 3-2 \text{ log. } 7-\text{ log. } 2-2 \text{ log. } 10=$$
$$2M\left[\frac{1}{19601}+\frac{1}{3}\left(\frac{1}{19601}\right)^3+\text{ \&c.}\right]$$

whence we have

$$\text{log. } 7=\text{ log. } 11+2 \text{ log. } 3-\tfrac{1}{2} \text{ log. } 2-1-M\left(\frac{1}{19601}+\text{ \&c.}\right)$$

Here the second term of the series is less than $\frac{1}{10^{13}}$; the first term will therefore suffice in finding the logarithm to eleven or twelve places. The same degree of accuracy is obtained with the use of only one term of the series in the following examples.

In (64) let $p+1=6656=2^9.13$, $p=6655=5.11^3$; then we have log. $(p+1)=9$ log. 2 + log. 13, log. $p=$ log. $5+3$ log. 11, and the formula gives

$$\text{log. } 13=\text{ log. } 5+3 \text{ log. } 11-9 \text{ log. } 2+\frac{2M}{13311}+\text{ \&c.}$$

Let $p+1=14400=2^4.3^2.10^2$, $p=14399=11^2.7.17$; then

$$\text{log. } 17=\text{ log. } 14400-\text{ log. } 847-\left(\frac{2M}{28799}+\text{ \&c.}\right)$$

Let $p+1=5776=19^2.2^4$, $p=5775=3.7.5^2.11$; then

$$\text{log. } 19=\tfrac{1}{2}(\text{log. } 5775-\text{ log. } 16)+\frac{M}{11551}+\text{ \&c.}$$

Let $p+1=8281=7^2.13^2$, $p=8280=2^3.3^2.5.23$; then

$$\text{log. } 23=\text{ log. } 8281-\text{ log. } 360-\left(\frac{2M}{16561}+\text{ \&c.}\right)$$

Let $p+1=13225=5^2.23^2$, $p=13224=2^3.3.19.29$; then

$$\text{log. } 29=\text{ log. } 13225-\text{ log. } 456-\left(\frac{2M}{26449}+\text{ \&c.}\right)$$

Let $p+1=8464=23^2.2^4$, $p=8463=3.7.13.31$; then

$$\text{log. } 31=\text{ log. } 8464-\text{ log. } 273-\left(\frac{2M}{16927}+\text{ \&c.}\right)$$

Let $p+1=5625=5^4.3^2$, $p=5424=2^3.19.37$; then

$$\log. 37=\log. 5625-\log. 152-\left(\frac{2M}{11249}+\&c.\right)$$

Let $p+1=6561=3^8$, $p=6560=2^5.5.41$; then

$$\log. 41=\log. 6561-\log. 160-\left(\frac{2M}{13121}+\&c.\right)^*$$

117. To continue the computation for primes above 41, we may employ *Borda's formula*, (65), without any transformations such as the above; and the first term will suffice in finding the logarithms to twelve places. Thus, to find log. 43 we make $p=41$, and find

$$\log. 43=2(\log. 42-\log. 40)+\log. 39+\frac{2M}{34399}+\&c.$$

In like manner

$$\log. 47=2(\log. 46-\log. 44)+\log. 43+\frac{2M}{45495}+\&c.$$

$$\log. 53=2(\log. 52-\log. 50)+\log. 49+\frac{2M}{66249}+\&c.$$

and with this formula we may compute the logarithms of all the prime numbers under 100.

Above 100 it will be more convenient to use (67), which requires only two logarithms to be known, and gives the logarithm correctly to twelve places with the use of only one term of the series. For example, when $q=103$, we have

$$\log. 103=\tfrac{1}{2}(\log. 102+\log. 104)+\frac{M}{21217}+\&c.$$

Log. 104 is known before log. 103, since 104 is a composite number $=8.13$. In general, q being a prime number, $q-1$, and $q+1$ must be composite, and their logarithms are found before that of q.

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[*] These transformations may be obtained, by inspection, from a table of composite numbers.

118. Above 500 the formulæ of Art. 110 may be employed, and the series omitted entirely except in computing very extended tables. Thus, in (70), if $n=500$ it is evident that $\frac{24}{n^5}$ is less than $\frac{1}{10^{12}}$; the series therefore will not affect the eleventh or twelfth place. Omitting the series, the formula may be put in the following form:

$$\log.(n+3)=\log.(n-2)+5[\log.(n+2)-\log.(n-1)]-10[\log.(n+1)-\log.n]. \qquad (74)$$

For example, the common logarithm of 509 may be found from the five logarithms immediately preceding it in the table, by a computation like the following, where we put

$$D=\log.(n+2)-\log.(n-1), \quad D'=\log.(n+1)-\log.n,$$

and the characteristics are omitted.

	N.	*Logarithms.*		*Calculation.*
$n-2$	504	70243,05364,45	. . . . .	70243,05364,45
$n-1$	505	70329,13781,19	$5D=$	1286,16708,25
n	506	70415,05168,40		71529,22072,70
$n+1$	507	70500,79593,33	$10D'=$	857,44249,30
$n+2$	508	70586,37122,84	$\log.(n+3)=$	70671,77823,40

When $n=10000$ in (68), we may omit the series, which will not affect the eleventh place of decimals; and we shall have

$$\log.(n+2)=\log.(n-1)-3[\log.(n+1)-\log.n.] \qquad (75)$$

Or we may omit the series in (65), and we shall have

$$\log.(p+2)=\log.(p-2)+2[\log.(p+1)-\log.(p-1.)] \qquad (76)$$

With these formulæ we may continue the computation as far as log. 100,000, which is the usual limit of the tables.

119. Above 100,000 we may use (67) without the series, if we require only ten places; we shall then compute by the formula

$$\log.q=\tfrac{1}{2}[\log.(q-1)+\log.(q+1)], \qquad (77)$$

from which it appears that the logarithm of any number above 100,000 is an arithmetical mean between the log. immediately preceding and that immediately succeeding it—at least as far as the tenth place of decimals. With this formula, then, we may extend the table at pleasure.

120. By retaining the series in the formulæ, and using one, two, or more terms, we may compute logarithms to twenty or more places, as has been done by Briggs and others.

The variety of the formulæ here given will enable the computist to test the correctness of his work by occasionally computing the same logarithm by two or more methods; he will thus detect the amount of the error occasioned by the neglected terms of the series, and may operate in such a manner that this error shall not affect the last figure retained.

ANTI-LOGARITHMS.

121. Formula (41) of Art. 91, enables us to compute the number corresponding to any given logarithm. We may thus obtain the *approximate numbers* corresponding to *exact logarithms*. These numbers have been called *anti-logarithms*, and tables of them are found convenient in some astronomical computations. In these tables the usual arrangement is reversed; the exact logarithms are placed first, increasing regularly by 1 from 1 to 10000, (or, as in Dodson's Anti-Logarithmic Canon, to 100,000,) and the corresponding nearest numbers in the columns opposite, with their differences and proportional parts.

122. In (41) let $b=\frac{m}{n}$, and $\log. b=\log. m-\log. n=D$; the formula becomes

$$m=n\left[1+\frac{D}{M}+\frac{1}{2}\left(\frac{D}{M}\right)^2+\frac{1}{2\,.\,3}\left(\frac{D}{M}\right)^3+\text{ \&c.}\right] \quad (78)$$

In applying this to naperian logarithms we make $M=1$; we then have

$$m=n\left(1+D+\frac{D^2}{2}+\frac{D^3}{2\,.\,3}+\text{ \&c.}\right) \quad (79)$$

If now we compute the anti-logs. of .00001, .00002, &c. sucessively, we shall have constantly $D=\log. m - \log. n = .00001$, and the formula will be

$$m=n\left(1+.00001+\frac{(.00001)^2}{2}+\frac{(.00001)^3}{2.3}+\&c.\right)$$

or $$m=n\times 1.00001,00000,5, \tag{80}$$

where m and n are any two numbers whose logarithms differ by .00001.

Let $\log. m = .00001$ and $\log. n = 0$, whence $n=1$, then m or the anti-logarithm of .00001 is found by (80) to be

$$\text{A. Log. }.00001 = 1.00001,00000,5.$$

Let $\log. m = .00002$, $\log. n = .00001$; then $n = \text{A. Log. }.00001 = 1.00001,00000,5$; and (80) gives

$$\text{A. Log. }.00002 = (1.00001,00000,5)^2.$$

We shall find in the same manner

$$\text{A. Log. }.00003 = (1.00001,00000,5)^3,$$
$$\text{A. Log. }.00004 = (1.00001,00000,5)^4,$$
$$\&c. \qquad \&c.$$

The construction of the table is thus reduced to finding the successive powers of the number 1.00001,00000,5, an operation of little difficulty. The calculation would be as follows:

$$\begin{array}{lr}
\text{A. Log. }.00001 = & 1.00001,00000,5 \\
 & 1,00001,0 \\
 & 5 \\
\hline
\text{A. Log. }.00002 = & 1.00002,00002,0 \\
 & 1,00002,0 \\
 & 5 \\
\hline
\text{A. Log. }.00003 = & 1.00003,00004,5 \\
 & 1,00003,0 \\
 & 5 \\
\hline
\text{A. Log. }.00004 = & 1.00004,00008,0
\end{array}$$

As this operation is carried on entirely by addition, the accumulated error arising from the neglected terms of the series may at length affect the last place of decimals retained. It will be proper, therefore, to calculate separately the anti-logs. of .00100, .00200, .00300, &c., which may be done by the method employed above. These will serve as test or proof numbers; and the intervening anti-logs. will be found by renewing the computation at .00100, .00200, &c. The error arising from the neglected figures will thus be corrected, and will not accumulate through more than 100 terms.

123. If we apply the formula to the common system, we make M=.434, &c., and by taking constantly D=.00001, as before, we shall have $\frac{D}{M}=.00002,30258,509$. The formula will become

$$m=n\times 1.00002,30261,16,$$

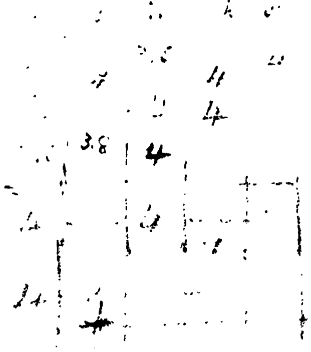
and we shall find as above

$$\text{A. Log. } .00001=1.00002,30261,16,$$
$$\text{A. Log. } .00002=(1.00002,30261,16)^2,$$
$$\text{A. Log. } .00003=(1.00002,30261,16)^3,$$
&c. &c.

For other methods of constructing anti-logarithms, the student is referred to Hutton's History of Logarithms, prefixed to the former editions of his "Mathematical Tables."

$$\frac{\frac{1}{3}\sqrt{2} \times \sqrt[3]{3}}{\sqrt{3} \times 2\sqrt[4]{2}} = \frac{\frac{1}{2}\sqrt{2}}{2\sqrt[4]{2} \times \sqrt{3}} \times \sqrt[3]{3} = \frac{\frac{1}{2}\sqrt{2}}{\sqrt[4]{32} \times \sqrt[2]{3}} \times \sqrt[3]{3} = \frac{\frac{1}{2}\sqrt{2}}{\sqrt{4\sqrt{2}} \times \sqrt{3}} \times \sqrt[3]{3} =$$

$$\frac{\frac{1}{2}\sqrt{2}}{\sqrt{12\sqrt{2}}} \times \sqrt[3]{3} = \frac{\sqrt{\frac{1}{4}\sqrt{4}}}{}$$



29.5

0.5

.05
.05
.0025